# Introduction to the Hyperasymptotic Technique in High-Frequency Computational Electromagnetics

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Abstract-Application of integral equation methods in computational electromagnetics has been widely explored in problems with varying degrees of complexity. Central to the formulation of such integral equation methods is the appropriate Green's function that in most cases is an important contributor to the accuracy of the final solution. It is however well known that at high frequencies special analytical forms of the problemmatched Green's function reduces the computation resources and hence renders solutions to electrically large problems practicable. These special high-frequency representations are derived analytically by well-known asymptotic methods when the characteristic wavenumber  $|\kappa| \to \infty$ . In this presentation a novel asymptotic method, known as hyperasymptotics, originally developed by Berry and Howls, is introduced. The main feature of the hyperasymptotic technique is that the numerical error in neglecting the remainder, obtained after optimal truncation of the asymptotic series, is of the order  $\mathcal{O}(e^{-C|\kappa|})$ , where C is a positive constant. Thus the error in the hyperasymptotic method decreases exponentially at high frequencies for  $|\kappa| \to \infty$ , and hence this specific asymptotic technique appears numerically most suitable for development of hybrid methods for challenging problems in computational electromagnetics. The salient features of the hyperasymptotic method is illustrated here with reference to the Stokes phenomenon for the Airy function of complex argument, and, its potential applications to some problems in computational electromagnetics are identified.

## I. INTRODUCTION

Problem-matched Green's function in hybrid integral equation [1]-[3] methods can increase the computational efficiency for electrically large problems and are obtained by asymptotic methods [4]-13]. For various applications [14]-[25] the numerical accuracy of the Green's function depends on how accurately its asymptotic form is obtained. This paper addresses the numerical accuracy of the various asymptotic methods [26]-[37], and in particular the Stokes phenomenon [38]-[44] that is intimately associated with the appropriate (optimal) termination of an infinite asymptotic series [45]-[47].

Recent investigations on *hyperasymptotics* [48]-[65] suggest that this new approach has the promise of providing numerically the most efficient form of asymptotic expansion of integrals that appear in the various Green's functions [66]-[70]. The hyperasymptotic method corrects the severe numerical errors occuring across the Stokes lines [57],[58].

In what follows, the Stokes phenomenon and hyperasymptotics is reviewed briefly from [48],[58] and [64] followed by an extensive list of references for the benefit of the reader.

#### **II. STOKES PHENOMENON**

The Stokes phenomenon can be described by starting with the one-dimensional (homogeneous) Sturm-Liouville problem [4, p. 274, Eq. (1)]

$$\frac{\mathrm{d}^2 g}{\mathrm{d}z^2} + P(z)\frac{\mathrm{d}g}{\mathrm{d}z} + Q(z)g = 0, \tag{1}$$

where the functions P, Q(z) are known a-priori and we may assume a propagation in the z-direction. In (1)  $g :\Longrightarrow g(z, z'; \kappa)$  maybe recognized as the one-dimensional transmission line green's function for an inhomogeneous media along the z-direction of propagation. Equation (1) can then be cast into the standard Wentzel-Kramers-Brillouin (WKB) form following the algebraic substitutions in [40, pp. 360-361] to obtain

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + w(z;\kappa)y = 0. \tag{2}$$

Millington [39] employed what is known as the Liouville-Green's transformation [40, ch. 14] that consists of transforming the both dependent and independent variables in (2), and obtained its following equivalent form

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \left[1 - \frac{\kappa}{x^2}\right]\Psi = 0,\tag{3}$$

to study the Stokes phenomenon. (It is worth noting that  $\kappa = \frac{-5}{36}$  in (3) further admits its reduction to the Airy's equation.)

When  $|x| \to \infty$ , the two elementary *asymptotic* solutions to (3) are  $\Psi^+(x) \approx e^{+jx}$  and  $\Psi^-(x) \approx e^{-jx}$ . A composite solution to (3) utilizing these two basic asymptotic forms may then be contructed that reads

$$\Psi(\kappa, x) \approx \mathcal{K}^+ \mathcal{M}^+(\kappa, x) e^{+\jmath x} + \mathcal{K}^- \mathcal{M}^-(\kappa, x) e^{-\jmath x}.$$
 (4)

In (4) the pre-factors  $\mathcal{M}^{\pm}(\kappa, x) = \exp[\mathcal{Q}^{\pm}(\kappa, x)]$ , where  $\mathcal{Q}^{\pm}(\kappa, |x| \to \infty) \to 0$  and  $\mathcal{K}^{\pm}$  are constants. Millington further showed through detailed analysis that the pre-factors  $\mathcal{M}^{\pm}(\kappa, x)$  are indeed multivalued, but remain bounded at  $|x| \to \infty$  in the complex x-plane.

This multivalued nature of  $\mathcal{M}^{\pm}(\kappa, x)$  is such that, say, in the upper half  $(\Im m(x) \ge 0)$  of the complex x-plane a radial branch cut may exist to uniquely define the prefactor  $\mathcal{M}^{+}(\kappa, x)$  but the other pre-factor  $\mathcal{M}^{-}(\kappa, x)$  could be single valued on the entire upper half plane. Similarly, on the lower half  $(\Im m(x) \leq 0)$  of the complex x-plane a radial branch cut may exist for preserving uniqueness of the pre-factor  $\mathcal{M}^{-}(\kappa, x)$  but the pre-factor  $\mathcal{M}^{+}(\kappa, x)$  could otherwise be very well be continuous (single-valued) there. Note also that in the u.h.p, the term  $e^{-jx}$  is *dominant* and  $e^{+jx}$  is *subdominant*, while in the l.h.p their respective roles are precisely reversed. For  $|x| \to \infty$ , since both the pre-factors  $\mathcal{M}^{\pm}(\kappa, |x| \to \infty) \to 1$ , one finds that the *numerical* character of  $\Psi(\kappa, x)$  is precisely dictated by the nature of the coefficients  $\mathcal{K}^{\pm}$  in (4), as  $|x| \to \infty$ .

Now, it may well happen that the original function  $\Psi(\kappa, x)$ in (3) is actually single valued everywhere and hence the asymptotic solution (for  $|x| \to \infty$ ) given in (4) simply cannot represent the actual function in the entire complex plane because of the inherent multivaluedness introduced in (4). Paraphrasing the precise description given by Paris [64, pp. 78-81], Stokes phenomenon is generally recognized to be a consequence of asymptotically representing an analytic function (with or without any multivalued structure) by approximants with a different multivalued structure.

Thus asymptotic expansion of integrals via the steepest descent methods, variously described in many well-known references [4]-[8],[28]-[36],[40]-[41],[58],[64], for the general case

$$\mathcal{I}(\kappa) = \int_{c} e^{\kappa w(\tau)} f(\tau) d\tau \tag{5}$$

when  $|\kappa| \to \infty$ , could very well lead to the Stokes phenomenon. This happens because upon locating the saddle point  $w'(\tau_s) = 0$  and the associated steepest descent paths emanating from  $\tau_s$ , the substitution  $s = w(\tau_s) - w(\tau)$  and its inversion via Lagrange's theorem [64, p. 11], for calculating the product  $f(\tau)\frac{ds}{d\tau}$ , needs to be carried out. This may complicate the nature of the integrand in (5) by introducing arbitrary multivalued forms in certain regions of the complex plane. This inversion process basically entails accurate calculation of the coefficients in the Laplace method [53]. Finally,  $\Lambda^{\pm}(\kappa, x) \equiv \mathcal{M}^{\pm}(\kappa, x) e^{\pm jx}$  in (4) may very well result from asymptotic expansions of integrals like (5) and hence lead to Stokes phenomenon involving numerical inaccuracies as discussed in [42]-[44],[47]-[52].

A qualitative study of the large argument behavior Airy function given by the integral

$$\operatorname{Ai}(z) = \frac{1}{2\pi \jmath} \int_{\mathsf{C}_1} e^{(z\tau - \frac{\tau^3}{3})} d\tau \tag{6}$$

which is an *exact* solution to the corresponding differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} - zf = 0,\tag{7}$$

obtained via contour integral transform [33, pp. 50-53], is a well-studied example [48] for illustrating the Stokes phenomenon.

The *composite* asymptotic expansion, as  $|z| \rightarrow \infty$  in (6), is given by two asymptotic series,  $u^+(z)$  and  $u^-(z)$ , respectively,

which from [34, p. 93, Eq. (4.38)] reads,

$$u^{\pm}(z) = \frac{z^{-\frac{1}{4}}e^{\pm\xi}}{2\pi} \sum_{n=0}^{+\infty} (\pm 1)^n \frac{c_n}{z^{3n/2}}, \text{ where,}$$
  

$$\xi = \frac{2}{3}z^{\frac{3}{2}} = |\xi|e^{+j\theta}, \text{ and,}$$
  

$$c_n = \frac{\Gamma(3n+\frac{1}{2})}{3^{2n}(2n)!}.$$
(8)

In view of the general composite asymptotic expansion in (4) the uniform asymptotic expansion to (6) can be written with the use of (8) as:

$$\operatorname{Ai}(z) \approx \mathcal{K}^{+} u^{+}(z) + \mathcal{K}^{-} u^{-}(z).$$
(9)

In (9) the constants

$$\begin{aligned} \mathcal{K}^+ &= 0, \text{ for } \frac{-\pi}{3} \le \operatorname{ph}(z) \le \frac{+\pi}{3}, \\ &= \jmath, \text{ for } \frac{\pi}{3} + \delta \le \operatorname{ph}(z) \le \frac{5\pi}{3} - \delta; \text{ and,} \\ \mathcal{K}^- &= 1, \text{ for } -\pi \le \operatorname{ph}(z) \le +\pi. \end{aligned}$$
(10)

The important observation from (10) is that the constant  $\mathcal{K}^+$  changes <u>discontinuously</u> as ph(z) changes continuously. This discontinuous change in constants across the Stokes lines is also known as the Stokes phenomenon. In his pioneering work, first reported in [48], Berry postulated based on the earlier work of Dingle [47] that this discontinuous change in constants in a compound asymptotic expansion is a very rapid but continuous change. Berry [48] starts with a general form closely similar to (4) to begin his analysis that reads,

$$\Psi(\kappa, x) \approx \mathcal{P}^{+}(\kappa, x) e^{\kappa \phi^{+}(x)} + \jmath \mathcal{S}(\kappa, x) \mathcal{P}^{-}(\kappa, x) e^{\kappa \phi^{-}(x)}.$$
(11)

In (11) the terms  $\mathcal{P}^+(\kappa, x)$  and  $\mathcal{P}^+(\kappa, x)$  are called *dominant* and *subdominant*, respectively. Berry, using the Borel summation formula [41, p. 382],[47, p. 406], shows that [48, Eqs. (20) & (21)]:

$$S(\kappa, x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\sigma} e^{-\chi^2} d\chi, \text{ where,}$$
  
$$\sigma = \kappa^{\frac{1}{2}} \frac{\Im m[\phi^+(x) - \phi^-(x)]}{2\Re e[\phi^+(x) - \phi^-(x)]}.$$
(12)

The Stokes lines where the discontinuity happens are those curves defined by  $\Im m[\phi^+(x) - \phi^-(x)] = 0$ . A more complete analysis of the Stokes lines is available in [57].

Next, the influence of the Stokes phenomenon in obtaining extremely accurate asymptotic expansion of the Airy function is outlined. The material below is primarily gleaned from [49]-[51] and [58],[64].

## **III. HYPERASYMPTOTICS**

The work of Berry [48], Berry and Howls [49]-[51] and Boyd [52] focusses on the numerical errors across the Stokes lines that are defined for the Airy function in (6) at  $ph(z) = \pm \frac{2}{3}\pi$ , or  $ph(\xi) = \pm \pi$  in view of (8). The *hyperasymptotic* form of the Airy function given below from [58] is for the Stokes line at  $ph(\xi) = \pi$ .

The main aspect of the hyperasymptotic expansion of the general integral given in (5) is that the resulting infinite asymptotic series is *truncated* at the just the smallest term, and the remainder of the asymptotic series is then evaluated via Borel summation which is then further asymptotically evaluated via the steepest descent method [50]. The analytical details are straightforward but quite complicated and good exposition can be found in [58]. The Airy function in (6) is now reexpressed [58, Eqs. (2.6,7)] along the steepest descent path,  $\Gamma_{\text{SDP}}$ , as:

Departing from the traditional use of the Borel transform for summing a divergent series as was utilized in [48, Eqs. (12)-(14)], the integral in (13) is finally expressed in [58] as

$$\mathcal{I}(\xi) = \sum_{n=0}^{N-1} c_n \xi^{-n} + R_N(\xi), \text{ where,}$$

$$R_N(\xi) = \frac{1}{2\pi} (-2\xi)^{-N} \int_0^{+\infty} e^{-x} x^{N-1} \left(1 + \frac{x}{2\xi}\right)^{-1} \mathcal{I}(x/2) dx.$$
(14)

In (14) the Laplace transform depicts a *resurgence* of the integral  $\mathcal{I}(x/2)$  given by (13). Subsequent analysis, using appropriate contour deformation, of the remainder  $R_N(\xi)$  shows that it can be bounded across the Stokes lines if the optimal truncation in (14) is chosen such that  $N = 2|\xi|$ , then with the constant  $C_2 = \sqrt{\pi}e^2C_1$ , where  $C_1$  is a real positive constant, the remainder is bounded by the condition [58, p. 515, Eq. (4.18)]

$$|\mathbf{R}_{\mathbf{N}}(\xi)| \le \mathcal{C}_2 e^{-2|\xi|}.$$
(15)

The result in (15) is the *main contribution* of the hyperasymptotic technique, resulting remainder to decay exponentially as the large argument  $|\xi| \rightarrow \infty$ . Thus, the remainder (or "higher order") terms in an asymptotic expansion are assured to decrease exponentially, which means that the hyperasymptotic expansion has the promise of delivering numerically the most accurate high-frequency (asymptotic) solution. This technique has not been used so far in high-frequency techniques for electrically large CEM problems.

Finally, the Airy function from [58, Eq. (5.6)] reads

$$\begin{aligned} \operatorname{Ai}(z) &\approx \frac{z^{-\frac{1}{4}}}{2\pi} \left(\frac{3}{2}\right)^{\frac{1}{2}} \left[ e^{-\xi} \sum_{p=0}^{2|\xi|-1} c_p \xi^{-p} \right. \\ &+ \frac{\jmath}{2} \operatorname{erfc}\{k(\theta)|\xi|^{\frac{1}{2}}\} e^{+\xi} \sum_{q=0}^{N'-1} (-1)^q c_q \xi^{-q} \right]. (16) \end{aligned}$$

In (13) to (16) the constants  $c_{n,p,q}$  are given in (8). In (16)  $erfc(\cdots)$  is the error function integral and is related to (12).

## IV. SUMMARY

In this brief exposition of the hyperasymptotic method the central argument advanced in favor of its application to electrically large CEM problems, is that this method has the promise of yielding the numerically most accurate highfrequency (asymptotic) Green's function. It is shown that a uniformization across the Stokes lines of the Airy function can be obtained that yields errors which decrease exponentially when the characteristic large parameter  $|\kappa| \rightarrow \infty$ . Numerical results to demonstrate the efficiency of the hyperasymptotic method will be provided at the time of the presentation.

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