Introduction to the Hyperasymptotic Technique in High-Frequency Computational Electromagnetics

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***Abstract*—Application of integral equation methods in computational electromagnetics has been widely explored in problems with varying degrees of complexity. Central to the formulation of such integral equation methods is the appropriate Green's function that in most cases is an important contributor to the accuracy of the final solution. It is however well known that at high frequencies special analytical forms of the problem-matched Green's function reduces the computation resources and hence renders solutions to electrically large problems practicable. These special high-frequency representations are derived analytically by well-known asymptotic methods when the characteristic wavenumber $|\kappa| \to \infty$. In this presentation a novel asymptotic method, known as *hyperasymptotics*, originally developed by Berry and Howls, is introduced. The main feature of the hyperasymptotic technique is that the numerical error in neglecting the remainder, obtained after optimal truncation of the asymptotic series, is of the order $\mathcal{O}(e^{-\mathrm{C}|\kappa|})$, where C is a positive constant. Thus the error in the hyperasymptotic method decreases exponentially at high frequencies for $|\kappa| \to \infty$, and hence this specific asymptotic technique appears numerically most suitable for development of *hybrid* methods for challenging problems in computational electromagnetics. The salient features of the hyperasymptotic method is illustrated here with reference to the Stokes phenomenon for the Airy function of complex argument, and, its potential applications to some problems in computational electromagnetics are identified.**

I. INTRODUCTION

Problem-matched Green's function in hybrid integral equation [1]-[3] methods can increase the computational efficiency for electrically large problems and are obtained by asymptotic methods [4]-13]. For various applications [14]-[25] the numerical accuracy of the Green's function depends on how accurately its asymptotic form is obtained. This paper addresses the numerical accuracy of the various asymptotic methods [26]-[37], and in particular the Stokes phenomenon [38]-[44] that is intimately associated with the appropriate (optimal) termination of an infinite asymptotic series [45]-[47].

Recent investigations on *hyperasymptotics* [48]-[65] suggest that this new approach has the promise of providing numerically the most efficient form of asymptotic expansion of integrals that appear in the various Green's functions [66]-[70]. The hyperasymptotic method corrects the severe numerical errors occuring across the Stokes lines [57],[58].

In what follows, the Stokes phenomenon and hyperasymptotics is reviewed briefly from [48],[58] and [64] followed by an extensive list of references for the benefit of the reader.

II. STOKES PHENOMENON

The Stokes phenomenon can be described by starting with the one-dimensional (homogeneous) Sturm-Liouville problem [4, p. 274, Eq. (1)]

$$\frac{\mathrm{d}^2 g}{\mathrm{d}z^2} + P(z)\frac{\mathrm{d}g}{\mathrm{d}z} + Q(z)g = 0, \tag{1}$$

where the functions $P, Q(z)$ are known a-priori and we may assume a propagation in the z-direction. In (1) $g :\Longrightarrow g(z, z^{'}; \kappa)$ maybe recognized as the one-dimensional transmission line green's function for an inhomogeneous media along the z-direction of propagation. Equation (1) can then be cast into the standard Wentzel-Kramers-Brillouin (WKB) form following the algebraic substitutions in [40, pp. 360-361] to obtain

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + w(z;\kappa)y = 0. \tag{2}$$

Millington [39] employed what is known as the Liouville-Green's transformation [40, ch. 14] that consists of transforming the both dependent and independent variables in (2), and obtained its following equivalent form

$$\frac{\mathrm{d}^2 \Psi}{\mathrm{d}x^2} + \left[1 - \frac{\kappa}{x^2}\right]\Psi = 0, \tag{3}$$

to study the Stokes phenomenon. (It is worth noting that $\kappa = \frac{-5}{36}$ in (3) further admits its reduction to the Airy's equation.)

When $|x| \to \infty$, the two elementary *asymptotic* solutions to (3) are $\Psi^{+}(x) \approx e^{+\jmath x}$ and $\Psi^{-}(x) \approx e^{-\jmath x}$. A composite solution to (3) utilizing these two basic asymptotic forms may then be contructed that reads

$$\Psi(\kappa, x) \approx \mathcal{K}^{+}\mathcal{M}^{+}(\kappa, x)e^{+\jmath x} + \mathcal{K}^{-}\mathcal{M}^{-}(\kappa, x)e^{-\jmath x}. \tag{4}$$

In (4) the pre-factors $\mathcal{M}^{\pm}(\kappa, x) = \exp[\mathcal{Q}^{\pm}(\kappa, x)]$, where $\mathcal{Q}^{\pm}(\kappa, |x| \to \infty) \to 0$ and $\mathcal{K}^{\pm}$ are constants. Millington further showed through detailed analysis that the pre-factors $\mathcal{M}^{\pm}(\kappa, x)$ are indeed multivalued, but remain bounded at $|x| \to \infty$ in the complex $x-$plane.

This multivalued nature of $\mathcal{M}^{\pm}(\kappa, x)$ is such that, say, in the upper half ($\Im m(x) \geq 0$) of the complex $x-$plane a radial branch cut may exist to uniquely define the pre-factor $\mathcal{M}^{+}(\kappa, x)$ but the other pre-factor $\mathcal{M}^{-}(\kappa, x)$ could be single valued on the entire upper half plane. Similarly,

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on the lower half ($\Im m(x) \le 0$) of the complex $x-$plane a radial branch cut may exist for preserving uniqueness of the pre-factor $\mathcal{M}^-(\kappa,x)$ but the pre-factor $\mathcal{M}^+(\kappa,x)$ could otherwise be very well be continuous (single-valued) there. Note also that in the u.h.p, the term $e^{-\jmath x}$ is *dominant* and $e^{+\jmath x}$ is *subdominant*, while in the l.h.p their respective roles are precisely reversed. For $|x|\to\infty$, since both the pre-factors $\mathcal{M}^{\pm}(\kappa,|x|\to\infty)\to 1$, one finds that the *numerical* character of $\Psi(\kappa,x)$ is precisely dictated by the nature of the coefficients $\mathcal{K}^{\pm}$ in (4), as $|x|\to\infty$.

Now, it may well happen that the original function $\Psi(\kappa,x)$ in (3) is actually single valued everywhere and hence the asymptotic solution (for $|x|\to\infty$) given in (4) simply cannot represent the actual function in the entire complex plane because of the inherent multivaluedness introduced in (4). Paraphrasing the precise description given by Paris [64, pp. 78-81], *Stokes phenomenon is generally recognized to be a consequence of asymptotically representing an analytic function* (*with or without any multivalued structure*) *by approximants with a different multivalued structure*.

Thus asymptotic expansion of integrals via the steepest descent methods, variously described in many well-known references [4]-[8],[28]-[36],[40]-[41],[58],[64], for the general case

$$\mathcal{I}(\kappa)=\int\limits_{\mathtt{C}} e^{\kappa w(\tau)}f(\tau)d\tau \tag{5}$$

when $|\kappa|\to\infty$, could very well lead to the Stokes phenomenon. This happens because upon locating the saddle point $w^{'}(\tau_s)=0$ and the associated steepest descent paths emanating from τ_s, the substitution $s=w(\tau_s)-w(\tau)$ and its inversion via Lagrange's theorem [64, p. 11], for calculating the product $f(\tau)\frac{ds}{d\tau}$, needs to be carried out. This may complicate the nature of the integrand in (5) by introducing arbitrary multivalued forms in certain regions of the complex plane. This inversion process basically entails accurate calculation of the coefficients in the Laplace method [53]. Finally, $\Lambda^{\pm}(\kappa,x)\equiv\mathcal{M}^{\pm}(\kappa,x)e^{\pm\jmath x}$ in (4) may very well result from asymptotic expansions of integrals like (5) and hence lead to Stokes phenomenon involving numerical inaccuracies as discussed in [42]-[44],[47]-[52].

A qualitative study of the large argument behavior Airy function given by the integral

$$\mathrm{Ai}(z)=\frac{1}{2\pi\jmath}\int\limits_{\mathtt{C}_1} e^{(z\tau-\frac{\tau^3}{3})}d\tau \tag{6}$$

which is an *exact* solution to the corresponding differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2}-zf=0, \tag{7}$$

obtained via contour integral transform [33, pp. 50-53], is a well-studied example [48] for illustrating the Stokes phenomenon.

The *composite* asymptotic expansion, as $|z|\to\infty$ in (6), is given by two asymptotic series, $u^+(z)$ and $u^-(z)$, respectively, which from [34, p. 93, Eq. (4.38)] reads,

$$\begin{aligned} u^{\pm}(z) &= \frac{z^{-\frac{1}{4}}e^{\pm\xi}}{2\pi}\sum_{n=0}^{+\infty}(\pm 1)^n\frac{c_n}{z^{3n/2}}, \text{ where,}\\ \xi &= \frac{2}{3}z^{\frac{3}{2}}=|\xi|e^{+\jmath\theta}, \text{ and,}\\ c_n &= \frac{\Gamma(3n+\frac{1}{2})}{3^{2n}(2n)!}. \end{aligned} \tag{8}$$

In view of the general composite asymptotic expansion in (4) the uniform asymptotic expansion to (6) can be written with the use of (8) as:

$$\mathrm{Ai}(z)\approx\mathcal{K}^+u^+(z)+\mathcal{K}^-u^-(z). \tag{9}$$

In (9) the constants

$$\begin{aligned} \mathcal{K}^+ &= 0, \text{ for } \frac{-\pi}{3}\le \mathrm{ph(z)}\le\frac{+\pi}{3},\\ &= \jmath, \text{ for } \frac{\pi}{3}+\delta\le\mathrm{ph(z)}\le\frac{5\pi}{3}-\delta; \text{ and,}\\ \mathcal{K}^- &= 1, \text{ for } -\pi\le\mathrm{ph(z)}\le+\pi. \end{aligned} \tag{10}$$

The important observation from (10) is that the constant $\mathcal{K}^+$ changes <u>*discontinuously*</u> as $\mathrm{ph(z)}$ changes continuously. *This discontinuous change in constants across the Stokes lines is also known as the Stokes phenomenon*. In his pioneering work, first reported in [48], Berry postulated based on the earlier work of Dingle [47] that this discontinuous change in constants in a compound asymptotic expansion is a *very rapid but continuous change*. Berry [48] starts with a general form closely similar to (4) to begin his analysis that reads,

$$\Psi(\kappa,x)\approx\mathcal{P}^+(\kappa,x)e^{\kappa\phi^+(x)}+\jmath\mathcal{S}(\kappa,x)\mathcal{P}^-(\kappa,x)e^{\kappa\phi^-(x)}. \tag{11}$$

In (11) the terms $\mathcal{P}^+(\kappa,x)$ and $\mathcal{P}^+(\kappa,x)$ are called *dominant* and *subdominant*, respectively. Berry, using the Borel summation formula [41, p. 382],[47, p. 406], shows that [48, Eqs. (20) & (21)]:

$$\begin{aligned} \mathcal{S}(\kappa,x) &= \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\sigma} e^{-\chi^2}d\chi, \text{ where,}\\ \sigma &= \kappa^{\frac{1}{2}}\frac{\Im m[\phi^+(x)-\phi^-(x)]}{2\Re e[\phi^+(x)-\phi^-(x)]}. \end{aligned} \tag{12}$$

The Stokes lines where the discontinuity happens are those curves defined by $\Im m[\phi^+(x)-\phi^-(x)]=0$. A more complete analysis of the Stokes lines is available in [57].

Next, the influence of the Stokes phenomenon in obtaining extremely accurate asymptotic expansion of the Airy function is outlined. The material below is primarily gleaned from [49]-[51] and [58],[64].

III. HYPERASYMPTOTICS

The work of Berry [48], Berry and Howls [49]-[51] and Boyd [52] focusses on the numerical errors across the Stokes lines that are defined for the Airy function in (6) at $\mathrm{ph}(z)=\pm\frac{2}{3}\pi$, or $\mathrm{ph}(\xi)=\pm\pi$ in view of (8). The *hyperasymptotic* form of the Airy function given below from [58] is for the Stokes line at $\mathrm{ph}(\xi)=\pi$.

The main aspect of the hyperasymptotic expansion of the general integral given in (5) is that the resulting infinite asymptotic series is *truncated* at the just the smallest term, and the remainder of the asymptotic series is then evaluated via Borel summation which is then further asymptotically evaluated via the steepest descent method [50]. The analytical details are straightforward but quite complicated and good exposition can be found in [58]. The Airy function in (6) is now reexpressed [58, Eqs. (2.6,7)] along the steepest descent path, Γ_{SDP}, as:

$$\begin{aligned}\mathrm{Ai}(z) &= \frac{1}{2\pi\jmath}\left(\frac{3}{2}\right)^{\frac{1}{3}}\xi^{-\frac{1}{6}}e^{-\xi}\mathcal{I}(\xi),\ \text{where},\\ \mathcal{I}(\xi) &= \xi^{\frac{1}{2}}\int_{\Gamma_{\mathrm{SDP}}} e^{\xi[f(u)+1]}du,\ \text{and},\\ f(u) &= \frac{1}{2}(u^3-3u).\end{aligned} \tag{13}$$

Departing from the traditional use of the Borel transform for summing a divergent series as was utilized in [48, Eqs. (12)-(14)], the integral in (13) is finally expressed in [58] as

$$\begin{aligned}\mathcal{I}(\xi) &= \sum_{n=0}^{\mathrm{N}-1}\mathrm{c}_n\xi^{-n}+\mathrm{R_N}(\xi),\ \text{where},\\ \mathrm{R_N}(\xi) &= \frac{1}{2\pi}(-2\xi)^{-\mathrm{N}}\int_0^{+\infty}e^{-x}x^{\mathrm{N}-1}\left(1+\frac{x}{2\xi}\right)^{-1}\mathcal{I}(x/2)dx.\end{aligned} \tag{14}$$

In (14) the Laplace transform depicts a *resurgence* of the integral $\mathcal{I}(x/2)$ given by (13). Subsequent analysis, using appropriate contour deformation, of the remainder $\mathrm{R_N}(\xi)$ shows that it can be bounded across the Stokes lines if the optimal truncation in (14) is chosen such that $\mathrm{N}=2|\xi|$, then with the constant $\mathcal{C}_2=\sqrt{\pi}e^2\mathcal{C}_1$, where $\mathcal{C}_1$ is a real positive constant, the remainder is bounded by the condition [58, p. 515, Eq. (4.18)]

$$\boxed{|\mathrm{R_N}(\xi)|\leq\mathcal{C}_2e^{-2|\xi|}.} \tag{15}$$

The result in (15) is the *main contribution* of the hyperasymptotic technique, resulting remainder to decay exponentially as the large argument $|\xi|\to\infty$. Thus, the remainder (or "higher order") terms in an asymptotic expansion are assured to decrease exponentially, which means that the hyperasymptotic expansion has the promise of delivering numerically the most accurate high-frequency (asymptotic) solution. This technique has not been used so far in high-frequency techniques for electrically large CEM problems.

Finally, the Airy function from [58, Eq. (5.6)] reads

$$\begin{aligned}\mathrm{Ai}(z) &\approx \frac{z^{-\frac{1}{4}}}{2\pi}\left(\frac{3}{2}\right)^{\frac{1}{2}}\left[e^{-\xi}\sum_{p=0}^{2|\xi|-1}\mathrm{c}_p\xi^{-p}\right.\\ &+ \left.\frac{\jmath}{2}\mathtt{erfc}\{k(\theta)|\xi|^{\frac{1}{2}}\}e^{+\xi}\sum_{q=0}^{\mathrm{N}'-1}(-1)^q\mathrm{c}_q\xi^{-q}\right].\end{aligned} \tag{16}$$

In (13) to (16) the constants $\mathrm{c}_{n,p,q}$ are given in (8). In (16) $\mathtt{erfc}(\cdots)$ is the error function integral and is related to (12).

IV. Summary

In this brief exposition of the hyperasymptotic method the central argument advanced in favor of its application to electrically large CEM problems, is that this method has the promise of yielding the numerically most accurate high-frequency (asymptotic) Green's function. It is shown that a uniformization across the Stokes lines of the Airy function can be obtained that yields errors which decrease exponentially when the characteristic large parameter $|\kappa|\to\infty$. Numerical results to demonstrate the efficiency of the hyperasymptotic method will be provided at the time of the presentation.

References

[1] E. H. Newman, "Overview of the Hybrid MM/GTD Green's Function Method in Electromagnetics," *Proc. of IEEE*, vol. 76, no. 3, pp. 270-282, March 1988.

[2] G. A. Thiele, "Overview of Selected Hybrid Methods in Radiating System Analysis," *Proc. of IEEE*, vol. 80, no. 1, pp. 66-78, January 1992.

[3] J. L. Volakis and K. Sertel, *Integral Equation Methods for Electromagnetics*. Raleigh, NC, USA: Scitech Publishing, Inc., 2012.

[4] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves* (Classic Reissue). NY, USA: IEEE Press, 1994.

[5] D. Bouche, F. Molinet and R. Mittra, *Asymptotic Methods in Electromagnetics*. NY, USA; Springer, 1997.

[6] M. Roseau, *Asymptotic Wave Theory*. NY, USA: North-Holland, 1976.

[7] J. R. Wait, *Electromagnetic Waves in Stratified Media* (classic reissue). NY, USA: IEEE Press, 1996.

[8] W. C. Chew, *Waves and Fields in Inhomogeneous Media*. NY, USA: IEEE Press, 1995.

[9] V. W. Hansen, *Numerical Solution of Antennas in Layered Media*. NY, USA: John Wiley and Sons, Inc., 1989.

[10] S. Barkeshli, P. H. Pathak and M. Marin, "An Asymptotic Closed-Form Microstrip Surface Green's Function for the Efficient Moment Method Analysis of Mutual Coupling in Microstrip Antennas," *IEEE Trans. Antennas. Propagat.*, vol. 38, no. 9, pp. 1374-1383, September 1990.

[11] L. Levey and L. B. Felsen, "On Transition Functions Occuring in the Theory of Diffraction in Inhomogeneous Media," *Jour. Inst. Math. Applics.* vol. 3, no. 1, pp. 76-97, 1967.

[12] D. V. Batorsky and L. B. Felsen, "Ray-Optical Calculation of Modes Excited by Sources and Scatterers in a Weakly Inhomogeneous Duct," *Radio Science*, vol. 6, no. 10, pp. 911-923, October 1971.

[13] V. A. Fock, *Electromagnetic Propagation and Diffraction Problems* . NY, USA: Pergamon Press, 1965.

[14] P. H. Pathak and N. N. Wang, "An Analysis of Mutual Coupling Between Antennas on a Smooth Convex Surface," final report 784583-7, The ElectroScience laboratory, Ohio State University, prepared under contract no. N66269-76-C-0554, Dept. of Navy, Naval Air Development Center, Warminster, PA, October 1978.

[15] J. Boersma and S. W. Lee, "Surface Field Due to a Magnetic Dipole on a Cylinder: Asymptotic Expansion of Exact Solution," tech. rep. 78-17, Electromagnetics Laboratory, University of Illinois at Urbana-Champaign, prepared under contract no. N00019-78-C-0064, Dept. of Navy, Naval Air Systems Command, Washington, DC, December 1978.

[16] S. W. Lee, “A Review of GTD Calculation of Mutual Admittance of Slot Conformal Array,” *Electromagnetics*, no. 2, pp. 85-127, 1982.

[17] L. Josefsson and P. Persson, *Conformal Array Antenna Theory and Design*. NY, USA: IEEE Press, Wiley-Interscience, 2006.

[18] P. H. Pathak, “An Asymptotic Analysis of the Scattering of Plane Waves by a Smooth Convex Cylinder”, *Radio Science*, vol. 14, no. 3, pp. 419-435, May-June 1979.

[19] P. H. Pathak, “Techniques for High-Frequency Problems,” chapter 4 in *Antenna Handbook: Theory, Applications and Design*, Y. T. Lo and S. W. Lee (eds.). NY, USA: Van Nostrand Reinhold Co., 1988.

[20] H.-T. Chou, P. H. Pathak and M. Hsu, “Extended Uniform Geometrical Theory of Diffraction Solution for Radiation by Antennas Located Close to Arbitrary, Smooth, Perfectly Conducting, Convex Surface,” *Radio Science* , vol. 32, no. 4, pp. 1297-1317, July-August 1997.

[21] G. N. Watson, “The Diffraction of Electric Waves by the Earth,” *Proc. Roy. Soc. Lond.*, ser. A, vol. 95, no. 666, pp. 83-99, October 1918.

[22] R. M. Lewis, N. Bleistein and D. Ludwig, “Uniform Asymptotic Theory of Creeping Waves,” *Comm. Pure. Appl. Math.*, vol. XX, pp. 295-328, 1967.

[23] F. G. Leppington, “Creeping Waves in the Shadow of an Elliptic Cylinder,” *Jour. Inst. Math. Applics* vol. 3, no. 1, pp. 388-402, 1967.

[24] S. Hong, “Asymptotic Theory of Electromagnetic and Acoustic Diffraction by Smooth Convex Surfaces of Variable Curvature,” *Jour. Math. Phys.*, vol. 8, no. 6, pp. 1223-1232, June 1967.

[25] R. H. Ott, “Scattering by a Parabolic Cylinder - A uniform asymptotic expansion,” *Jour. Math . Phys.*, vol. 26, no. 4, pp. 854-860, April 1985.

[26] E. W. Barnes, “The Asymptotic Expansion of Integral Functions defined by Taylor Series,” *Proc. Roy. Soc. Lond.*, ser. A, vol. 206, pp. 249-297, 1906.

[27] G. N. Watson, “A Theory of Asymptotic Series,” *Proc. Roy. Soc. Lond.*, ser. A, vol. 211, pp. 279-313, 1912.

[28] A. Erdélyi, *Asymptotic Expansions*. NY, USA: Dover, 1956.

[29] E. T. Copson, *Asymptotic Expansions*. NY, USA: Cambridge University Press, 1965.

[30] B. L. van der Waerden, “On the Method of Saddle Points,” *Applied Scientific Research*, part B, vol. 2, issue 1, pp. 33-45, December 1952.

[31] P. C. Clemmow, “Some Extensions to the Method of Integration by Steepest Descents,” *Quarterly Journal of Mechanics and Applied Mathematics*, vol. III, part 2, pp. 241-256, 1950.

[32] M. J. Ablowitz and A. S. Fokas, *Complex Variables: Introduction and Applications*. NY, USA: Camridge University Press, 1997.

[33] N. Bleistein and R. A Handelsman, *Asymptotic Expansions of Integrals*. NY, USA: Dover, 1986.

[34] R. Wong, *Asymptotic Approximations of Integrals*. PA, USA: SIAM, 2001.

[35] A. Ciarkowski, “Uniform Asymptotic Expansion of an Integral with Saddle Point, Pole and a Branch Point,” *Proc. Roy. Soc. Lond.*, ser. A, vol. 426, pp. 273-286, 1989.

[36] N. M. Temme, “Uniform Asymptotic Expansion of Integrals: a selection of problems,” *Journal of Computational and Applied Mathematics* (JCAM), vol. 65, issue 1-3, pp. 395-417, December 1995.

[37] R. Y. S. Lynn and J. B. Keller, “Uniform Asymptotic Solution of Second Order Linear Differential Equations with Turning Points,” *Communications on Pure and Applied Mathematics*, vol. XXIII, pp. 379-408, 1970.

[38] J. Heading, *An Introduction to Phase Integral Methods*. NY, USA: John-Wiley & Sons, Inc., 1962.

[39] G. Millington, “Stokes Phenomenon,” *Radio Science*, vol. 4, no. 2, pp. 95-115, February 1969.

[40] A. H. Nayfeh, *Introduction to Perturbation Techniques*. NY, USA: Wiley-InterScience, John-Wiley & Sons, Inc., 1993 (Wiley Classic Publication).

[41] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory* I. NY, USA: Springer, 1999.0

[42] R. B. White, *Asymptotic Analysis of Differential Equations*. NJ, USA: Imperial College Press/World Scientific, 2010.

[43] M. V. Fedoryuk, *Asymptotic Analysis: Linear Ordinary Differential Equations*. NY, USA: Springer-Verlag, 1993.

[44] N. Fröman and P. O. Fröman, *Physical Problems Solved by the Phase-Integral Method*. NY, USA: Cambridge University Press, 2004.

[45] G. M. Roe, “An Accurate Method for Terminating Asymptotic Series,” SIAM *Jour. Appl. Math.*, vol. 8, no. 2, pp. 354-367, June 1960.

[46] W. B. Ford, *Studies on Divergent Series and Summability and The Asymptotic Developments of Functions Defined by Maclaurin’s Series*. NY, USA: Chelsea Pub. Co., 1960.

[47] R. B. Dingle, *Asymptotic Expansions: Their Interpretation and Derivation*. NY, USA: Academic Press, 1973.

[48] M. V. Berry, “Uniform Asymptotic Smoothing of Stokes Discontinuities,” *Proc. Roy. Soc. Lond.*, ser. A, vol. 422, no. 1862, pp. 7-21, March 1989.

[49] M. V. Berry and C. J. Howls, “Hyperasymptotics,” *Proc. Roy. Soc. Lond.*, ser. A, vol. 430, pp. 653-667, 1990.

[50] M. V. Berry and C. J. Howls, “Hyperasymptotics for Integrals with Saddles,” *Proc. Roy. Soc. Lond.*, ser. A, vol. 434, pp. 657-675, 1991.

[51] C. J. Howls, “Hyperasymptotics for Integrals with Finite Endpoints,” *Proc. Roy. Soc. Lond.*, ser. A, vol. 439, no. 1906, pp. 373-396, Nov. 1992.

[52] W. G. C. Boyd, “Error Bounds for the Method of Steepest Descents,” *Proc. Roy. Soc. Lond.*, ser. A, vol. 440, no. 1910, pp. 493-518, March 1993.

[53] J. Wojdylo, “Computing the Coefficients in Laplace’s Method,” SIAM *Review*, vol. 48, no. 1, pp. 76-96, March 2006.

[54] A. B. Olde-Daalhuis, S. J. Chapman, J. R. King, J. R. Ockendon and R. H. Tew, “Stokes Phenomenon and Matched Asymptotic Expansions,” SIAM *Jour. Appl. Math.*, vol. 65, no. 6, pp. 1469-1483, December 1995.

[55] J. G. B. Byatt-Smith, “Formulation and Summation of Hyperasymptotic Expansions obtained from Integrals,” *Eur. Jour. Appl. Math.*, vol. 9, no. 2, pp. 159-185, April 1998.

[56] J. P. Boyd, “The Devil’s Invention: Asymptotic, Superasymptotic and Hyperasymptotic Series,” *Acta Applicande Mathematicae*, no. 56, pp. 1-98, 1999.

[57] T. Aoki, T. Kawai and Y. Takei, “On the Exact Steepest Descent Method: A New Method of Description of Stokes Curves,” *Jour. Math. Phys.*, vol. 42, no. 8, pp. 3691-3713, August 2001.

[58] R. Wong, “Exponential Asymptotics,” in *Special Functions* 2000*: Current Perspectives and Future Directions*, J. Bustoz, M. E. H. Ismail and S. K. Suslov (eds.). Dordrecht, Boston, USA: Kluwer Academic Publishers, 2001.

[59] V. Kowalenko, “Exactification of the Asymptotics for Bessel and Hankel Functions,” *Appl. Math. Comp.*, no. 133, pp. 487-518, 2002.

[60] J. B. Lopéz, P. Pagola, E. P. Sinsuía, “A Systemization of the Saddle Point Method. Application to Airy and Hankel Functions,” *Jour. Math. Anal. Appl.*, vol. 354, pp. 347-359, 2009.

[61] E. J. Weniger, “Summation of Divergent Power Series by means of Factorial Series,” *Appl. Numer. Math.*, no. 60, pp. 1429-1441, 2010.

[62] R. Borghi, “Joint Use of Weniger Transformation and Hyperasymptotics for Accurate Asymptotic Evaluations of a Class of Saddle-Point Integrals,” *Physical Review* E, no. 78, pp. 026703-1 to 11, 2008.

[63] R. Borghi, “Joint Use of Weniger Transformation and Hyperasymptotics for Accurate Asymptotic Evaluations of a Class of Saddle-Point Integrals II: Higher Order Transformations,” *Physical Review* E, no. 80, pp. 016704-1 to 15, 2009.

[64] R. B. Paris, *Hadamard Expansions and Hyperasymptotic Evaluation: An Extension of the Method of Steepest Descents*. NY, USA: Cambridge University Press, 2011.

[65] J. B. Lopéz and P. J. Pagola, “A Systematic “Saddle-point-near-a-Pole” Asymptotic Method with Application to Gauss Hypergeometric Function”, *Stud. in Appl. Math.*, vol. 127, pp. 24-37, 2011.

[66] G. Caviglia and A. Morro, “First Integrals and Turning Points for Wave Propagation in Planarly-Stratified Media,” *Acta Mechanica*, vol. 169, pp. 1-11, May 2004. (online)

[67] S. Ancey, A. Folacci and P. Gabrielli, “Exponentially Improved Asymptotic Expansions for Resonances of an Elliptic Cylinder,” *Jour. Phys.* A: *Math. Gen.*, vol. 33, pp. 3179-3208, 2000.

[68] Y. Umul, “The Physical Optics Integral of Creeping Waves,” *Jour. Electromag. Waves. Applics.*, vol. 26, pp. 1203-1214, 2012.

[69] J. F. Legendre, T. Marsault, G. Landouer and D. Cueff, “Fock’s Integral and Pekeris’s Integral Calculation for Diffraction on Dielectric Convex Surface,” *Microw. Opt. Tech. Lett.* (MoTEL), vol. 54, no. 11, pp. 2464-2468, November 2012.

[70] D. Chatterjee, “Phase-Integral Formulation of the Single-Layer Microstrip Green’s Function,”*Proceedings URSI* (Commission B) *Symposium on Electromagnetic Theory*, pp. 1085-1088, Berlin, Germany, August 16-19, 2010.