Stored energies for electric and magnetic currents with applications to Q for small antennas

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***Abstract*—Fundamental limitations restrict the bandwidth of electrically small antennas. A method to obtain such limitations is based on stored electric and magnetic energies. Vandenbosch recently proposed a set of integral expression for the stored energies. These expressions provide a method to determine optimal currents and physical bounds on antennas for maximal bandwidth and desired radiated fields. In the present paper, we generalize the expressions for the stored energies to include magnetic sources for small structures. We give an expression for the antenna Q for electrically small shapes. Examples for small spheres are tested and the results agree with the published results.**

I. Introduction

Many new results in fundamental limitation on antennas have been proposed in the last decade, see *e.g.*, [1–14]. These approaches generalize the well-known Chu bound [15] on the antenna Q, Q, also called or radiation Q, that is based on spherical harmonics to determine stored and radiated energies outside a small sphere enclosing the antenna [15]. The use of the forward scattering sum rule in [5, 6, 8, 10] broke the dependence on generic shapes such as Chu's circumscribing sphere and presents bounds for arbitrary shaped structures. Lower bounds on Q form different starting points are also presented in [7, 9, 16].

The electric and magnetic energy in the fields are decomposed into stored electric and magnetic energies and radiated energy. This decomposition is non-unique, and different choices are possible see *e.g.*, [17, 18] for implications of such definitions and discussion see [13]. Vandenbosch proposed a decomposition in [17] that is interesting since it provides an explicit integral expression of the stored energies in the electric current densities. Fundamental bounds based on these energies are derived in [11–13, 16]. It is also shown that these bounds agree well with the results from the forward scattering sum-rule [5, 6, 8, 10] for small and intermediate sized antennas. One benefit of the energy expression in [13, 17] is that many antenna problems can be formulated as convex optimization problems that determines realizations of current densities that achieve the optimal behavior [12]. Thin-wire and modal applications of the energies are given in [19, 20]. In [11], it is shown that the energies can become negative for currents on larger volumes, and in [13] we present an alternative way to derive the energies for electric sources.

The formulation as presented by Vandenbosch extends to rather general linear media, however the obtained results in [17] are expressed in quadratic terms of a *generalized* current containing cross terms and derivatives of the original currents in a non-trivial way. In this paper we derive the stored energies in terms of the electric and magnetic currents to leading order for small antenna structures, and the dual symmetry between electric and magnetic current densities is apparent. It is shown that the leading order terms of these energies are non-negative in the limit of small antennas. In addition we derive the radiated power density for both electric and magnetic sources, and combine them to obtain the leading order behavior of antenna Q for small volumes. Using basic properties of the currents, we derive the lowest order behavior of Q. We show that these results, when they are applied to a sphere agree with the electrical case for a sphere considered by *e.g.*, Vandenbosch [16, 17], more general current configurations are also presented.

II. Stored energies

The equivalence theorems of Huygens, Love and Schelkunoff [21, 22] shows that it is advantageous to introduce equivalent magnetic current and charge densities $\boldsymbol{J}_{\mathrm{m}}$, ρ_{m} in addition to the electric current and charge densities $\boldsymbol{J}_{\mathrm{e}}$, ρ_{e} in order to understand the behavior of electromagnetic fields. Here, we let all these source terms have support in V, where $V \subset \mathbb{R}^3$ is a smooth and bounded domain. We assume that the source terms are sufficiently smooth functions of space and that the field are in free space. For this case Maxwell's equation then takes the form

$$\nabla\times\boldsymbol{E} = -\mathrm{j}\eta_0 k\boldsymbol{H} - \boldsymbol{J}_{\mathrm{m}},\quad \nabla\times\boldsymbol{H} = \boldsymbol{J}_{\mathrm{e}} + \frac{\mathrm{j}k}{\eta_0}\boldsymbol{E}, \tag{1}$$

$$\nabla\cdot\boldsymbol{E} = \frac{\rho_{\mathrm{e}}}{\varepsilon_0} = \frac{-\eta_0}{\mathrm{j}k}\nabla\cdot\boldsymbol{J}_{\mathrm{e}},\quad \nabla\cdot\boldsymbol{H} = \frac{\rho_{\mathrm{m}}}{\mu_0} = \frac{-1}{\mathrm{j}k\eta_0}\nabla\cdot\boldsymbol{J}_{\mathrm{m}}. \tag{2}$$

As usual $\boldsymbol{E}$, $\boldsymbol{H}$ are the electric and magnetic fields respective and $\eta_0, \varepsilon_0, \mu_0$ are the impedance, permittivity and permeability of free space respectively, k is the wave number. Here we have used the time-convention $\mathrm{e}^{\mathrm{j}\omega t}$. Below we present stored reactive energies of Vandenbosch type. First the case of electrical sources, derived by Vandenbosch [17] and derive the electrical small behavior, that is $k \to 0$. We further extend this to include the purely magnetic case. The lowest order behavior for the combined case is also given.

A. Stored energies for electrical sources

The Vandenbosh expressions for the stored energies for electric sources, $\boldsymbol{J}_\mathrm{m} = 0$, $\rho_\mathrm{m} = 0$ are presented in [17]. The electric stored energy is $W_\mathrm{E} = \frac{\mu_0}{16\pi k^2} w_\mathrm{E}$, where the reduced energy w_E is

$$w_\mathrm{E} = \int_V \int_V \nabla_1 \cdot \boldsymbol{J}_\mathrm{e1} \nabla_2 \cdot \boldsymbol{J}^*_\mathrm{e2} \frac{\cos(kr_{12})}{r_{12}} - \frac{k}{2}\big(k^2 \boldsymbol{J}_\mathrm{e1} \cdot \boldsymbol{J}^*_\mathrm{e2} - \nabla_1 \cdot \boldsymbol{J}_\mathrm{e1} \nabla_2 \cdot \boldsymbol{J}^*_\mathrm{e2}\big) \sin(kr_{12}) \, \mathrm{d}V_1 \, \mathrm{d}V_2, \quad (3)$$

and where $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$, $\boldsymbol{J}_\mathrm{e1} = \boldsymbol{J}_\mathrm{e}(\boldsymbol{r}_1)$, $\boldsymbol{J}_\mathrm{e2} = \boldsymbol{J}_\mathrm{e}(\boldsymbol{r}_2)$. Magnetic stored energy is $W_\mathrm{M} = \frac{\mu_0}{16\pi k^2} w_\mathrm{M}$, where

$$w_\mathrm{M} = \int_V \int_V k^2 \boldsymbol{J}_\mathrm{e1} \cdot \boldsymbol{J}^*_\mathrm{e2} \frac{\cos(kr_{12})}{r_{12}} - \frac{k}{2}\big(k^2 \boldsymbol{J}_\mathrm{e1} \cdot \boldsymbol{J}^*_\mathrm{e2} - \nabla_1 \cdot \boldsymbol{J}_\mathrm{e1} \nabla_2 \cdot \boldsymbol{J}^*_\mathrm{e2}\big) \sin(kr_{12}) \, \mathrm{d}V_1 \, \mathrm{d}V_2, \quad (4)$$

see also [2].

To derive the small k-case we begin with the assumption that $\boldsymbol{J}_\mathrm{e} = \boldsymbol{J}^{(0)}_\mathrm{e} + k\boldsymbol{J}^{(1)}_\mathrm{e} + \dots$ with $\nabla \cdot \boldsymbol{J}^{(0)}_\mathrm{e} = 0$ and $\nabla \cdot \boldsymbol{J}^{(1)}_\mathrm{e} = -\mathrm{j}\rho^{(1)}_\mathrm{e}/\sqrt{\varepsilon_0\mu_0}$. Inserting this into the above energies reduces them to leading order as:

$$W_\mathrm{E} = \frac{1}{16\pi\varepsilon_0} \int_V \int_V \frac{\rho^{(1)}_\mathrm{e}(\boldsymbol{r}_1)\rho^{(1)*}_\mathrm{e}(\boldsymbol{r}_2)}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \, \mathrm{d}V_1 \, \mathrm{d}V_2 = \frac{1}{4}\int_V \Phi \rho^{(1)*}_\mathrm{e} \, \mathrm{d}V \quad (5)$$

$$W_\mathrm{M} = \frac{\mu_0}{16\pi} \int_V \int_V \frac{\boldsymbol{J}^{(0)}_\mathrm{e}(\boldsymbol{r}_1) \cdot \boldsymbol{J}^{(0)*}_\mathrm{e}(\boldsymbol{r}_2)}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \, \mathrm{d}V_1 \, \mathrm{d}V_2 = \frac{1}{4}\int_V \boldsymbol{A} \cdot \boldsymbol{J}^{(0)*}_\mathrm{e} \, \mathrm{d}V \quad (6)$$

Here $\boldsymbol{A}$ and Φ are the electric vector and scalar potential. Note that the stored energies correspond to half the electro-static and magneto-static energy. It is clear that these energies are non-negative. This result is also given in [11]. The equations (3)-(4) are also the leading order terms in [13].

B. Stored energies for magnetic sources

To derive the stored energies from purely magnetic sources, *i.e.*, when $\boldsymbol{J}_\mathrm{e} = 0$ and $\rho_\mathrm{e} = 0$ we use the electric-magnetic duality [23]. Maxwell's equations remain invariant under the duality interchange $(\boldsymbol{E}_\mathrm{e}, \eta_0\boldsymbol{H}_\mathrm{e}, \eta_0\boldsymbol{J}_\mathrm{e}, \eta_0\rho_\mathrm{e}) \mapsto (-\eta_0\boldsymbol{H}_\mathrm{m}, \boldsymbol{E}_\mathrm{m}, -\boldsymbol{J}_\mathrm{m}, \rho_\mathrm{m})$. The case with zero magnetic sources turns into the case of zero electric sources by this duality exchange which results in that the reduced energies satisfy:

$$\eta_0^2 w_\mathrm{E} = \int_V \int_V k^2 \boldsymbol{J}_\mathrm{m1} \cdot \boldsymbol{J}^*_\mathrm{m2} \frac{\cos(kr_{12})}{r_{12}} - \frac{k}{2}\big(k^2 \boldsymbol{J}_\mathrm{m1} \cdot \boldsymbol{J}^*_\mathrm{m2} - \nabla_1 \cdot \boldsymbol{J}_\mathrm{m1} \nabla_2 \cdot \boldsymbol{J}^*_\mathrm{m2}\big) \sin(kr_{12}) \, \mathrm{d}V_1 \, \mathrm{d}V_2, \quad (7)$$

where $\boldsymbol{J}_\mathrm{m1} = \boldsymbol{J}_\mathrm{m}(\boldsymbol{r}_1)$, $\boldsymbol{J}_\mathrm{m2} = \boldsymbol{J}_\mathrm{m}(\boldsymbol{r}_2)$. Furthermore,

$$\eta_0^2 w_\mathrm{M} = \int_V \int_V \nabla_1 \cdot \boldsymbol{J}_\mathrm{m1} \nabla_2 \cdot \boldsymbol{J}^*_\mathrm{m2} \frac{\cos(kr_{12})}{r_{12}} - \frac{k}{2}\big(k^2 \boldsymbol{J}_\mathrm{m1} \cdot \boldsymbol{J}^*_\mathrm{m2} - \nabla_1 \cdot \boldsymbol{J}_\mathrm{m1} \nabla_2 \cdot \boldsymbol{J}^*_\mathrm{m2}\big) \sin(kr_{12}) \, \mathrm{d}V_1 \, \mathrm{d}V_2, \quad (8)$$

To derive the electrically small case we assume that $\boldsymbol{J}_\mathrm{m} = \boldsymbol{J}^{(0)}_\mathrm{m} + k\boldsymbol{J}^{(1)}_\mathrm{m} + \dots$ with $\nabla \cdot \boldsymbol{J}^{(0)}_\mathrm{m} = 0$ and $\nabla \cdot \boldsymbol{J}^{(1)}_\mathrm{m} = -\mathrm{j}\rho^{(1)}_\mathrm{m}/\sqrt{\varepsilon_0\mu_0}$ that follows from the continuity equation for magnetic sources. Inserting this approximation of the sources into the reduced energies results to leading order to:

$$W_\mathrm{E} = \frac{\varepsilon_0}{16\pi} \int_V \int_V \frac{\boldsymbol{J}^{(0)}_\mathrm{m}(\boldsymbol{r}_1) \cdot \boldsymbol{J}^{(0)*}_\mathrm{m}(\boldsymbol{r}_2)}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \, \mathrm{d}V_1 \, \mathrm{d}V_2 = \frac{1}{4}\int_V \boldsymbol{F} \cdot \boldsymbol{J}^{(0)*}_\mathrm{m} \, \mathrm{d}V \quad (9)$$

$$W_\mathrm{M} = \frac{1}{16\pi\mu_0} \int_V \int_V \frac{\rho^{(1)}_\mathrm{m}(\boldsymbol{r}_1)\rho^{(1)*}_\mathrm{m}(\boldsymbol{r}_2)}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \, \mathrm{d}V_1 \, \mathrm{d}V_2 = \frac{1}{4}\int_V \Psi \rho^{(1)*}_\mathrm{m} \, \mathrm{d}V \quad (10)$$

Here Ψ and $\boldsymbol{F}$ is the static magnetic scalar and vector potential respectively. These energies are also clearly non-negative.

C. Stored energy for both electric and magnetic sources

The stored reactive electric and magnetic energies for the general case are quite involved and require a lengthy derivation; however for the special case of electrical small volumes we can simplify the results. Starting from the energy densities appearing in Poynting's theorem we can investigate the behavior of the cross terms by a detailed small k analysis of $\mathrm{Im}(\boldsymbol{E} \cdot \boldsymbol{J}^*_\mathrm{e} + \boldsymbol{H}^* \cdot \boldsymbol{J}_\mathrm{m})$. Investigation of this expression with help of Green's function and the expansions of $\boldsymbol{J}_\mathrm{e}$ and $\boldsymbol{J}_\mathrm{m}$ in orders of k shows that the cross term have a higher order k-dependence than the leading quadratic electric and magnetic sources. Thus, to leading order in k we note a decoupling of the sources for the stored energies. Consequently, we find that the leading order stored energies are:

$$W_\mathrm{E} = \frac{1}{16\pi} \int_V \int_V \big(\varepsilon_0 \boldsymbol{J}^{(0)}_\mathrm{m1} \cdot \boldsymbol{J}^{(0)*}_\mathrm{m2} + \frac{1}{\varepsilon_0}\rho^{(1)}_\mathrm{e1}\rho^{(1)*}_\mathrm{e2}\big)\frac{1}{r_{12}} \, \mathrm{d}V_1 \, \mathrm{d}V_2 = \frac{1}{4}\int_V \Phi\rho^{(1)}_\mathrm{e} + \boldsymbol{F} \cdot \boldsymbol{J}^{(0)}_\mathrm{m} \, \mathrm{d}V, \quad (11)$$

$$W_\mathrm{M} = \frac{1}{16\pi} \int_V \int_V \big(\mu_0 \boldsymbol{J}^{(0)}_\mathrm{e1} \cdot \boldsymbol{J}^{(0)*}_\mathrm{e2} + \frac{1}{\mu_0}\rho^{(1)}_\mathrm{m1}\rho^{(1)}_\mathrm{m2}\big)\frac{1}{r_{12}} \, \mathrm{d}V_1 \, \mathrm{d}V_2 = \frac{1}{4}\int_V \boldsymbol{A} \cdot \boldsymbol{J}^{(0)}_\mathrm{e} + \Psi\rho^{(1)}_\mathrm{m} \, \mathrm{d}V, \quad (12)$$

where $\rho_{e1} = \rho_e(\boldsymbol{r}_1)$ and similar for ρ_{e2}, ρ_{m1} and ρ_{m2}. This is the sum of the electric and the magnetic case. Since each of the terms is non-negative, the stored energies remain non-negative in the $k \to 0$ limit.

III. Radiated power

Starting from Poynting's theorem we have the radiated power density given by

$$\frac{1}{2}\operatorname{Re}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}^*) = \frac{-1}{2}\operatorname{Re}(\boldsymbol{E}\cdot\boldsymbol{J}_e^* + \boldsymbol{H}^*\cdot\boldsymbol{J}_m) \quad (13)$$

Insert Helmholtz Green's function to express the fields in terms of their sources. Extensive simplification utilizing partial integration and standard vector identities results in the radiated power:

$$\begin{aligned} P_r = \frac{1}{2}\operatorname{Re}\int_V \nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}^*)\,dV = \\ \int_V\int_V \frac{\eta_0}{2k}\Big[k^2\boldsymbol{J}_{e1}\cdot\boldsymbol{J}_{e2}^* - (\nabla\cdot\boldsymbol{J}_{e1})(\nabla\cdot\boldsymbol{J}_{e2}^*) \\ + \frac{1}{\eta_0^2}(k^2\boldsymbol{J}_{m1}^*\cdot\boldsymbol{J}_{m2} - (\nabla\cdot\boldsymbol{J}_{m1}^*)(\nabla\cdot\boldsymbol{J}_{m2}))\Big]\frac{\sin(kr_{12})}{4\pi r_{12}} \\ + \frac{k^2}{4\pi}\mathrm{j}_1(kr_{12})\hat{\boldsymbol{r}}_{12}\cdot\operatorname{Im}(\boldsymbol{J}_{e1}^*\times\boldsymbol{J}_{m2})\,dV_1\,dV_2 \quad (14) \end{aligned}$$

Here j_1 is the spherical Bessel function of first order.

We consider the limit $k \to 0$. Note from *e.g.*, [24] that $\int_V \boldsymbol{J}_{e,m}^{(1)}\,dV = \mathrm{j}\boldsymbol{p}_{e,m}/(\sqrt{\varepsilon_0\mu_0})$ and $\int_V(\hat{\boldsymbol{r}}_2\cdot\boldsymbol{r})\boldsymbol{J}_{e,m}^{(0)}\,dV = \boldsymbol{m}_{e,m}\times\hat{\boldsymbol{r}}_2$ since $\nabla\cdot\boldsymbol{J}_{e,m} = 0$. Here $\boldsymbol{J}_{e,m}$ indicate that the result both for the electric and the magnetic current density, similarly for magnetic dipole moment $\boldsymbol{m}$ and the electric dipole moment $\boldsymbol{p}$: $\boldsymbol{p}_{e,m} = \int_V \boldsymbol{r}\rho_{e,m}^{(1)}\,dV$ and $\boldsymbol{m}_{e,m} = \frac{1}{2}\int_V \boldsymbol{r}\times\boldsymbol{J}_{e,m}^{(0)}\,dV$. Upon utilizing that $\int_V \nabla\cdot\boldsymbol{J}_{e,m}^{(j)}\,dV = 0$ and $\int_V \boldsymbol{J}_{e,m}^{(0)}\,dV = 0$ we find that

$$P_r = \frac{\omega k^3}{12\pi}\Big[\big|\frac{1}{\sqrt{\varepsilon_0}}\boldsymbol{p}_e - \sqrt{\varepsilon_0}\boldsymbol{m}_m\big|^2 + \big|\frac{1}{\sqrt{\mu_0}}\boldsymbol{p}_m + \sqrt{\mu_0}\boldsymbol{m}_e\big|^2\Big] \quad (15)$$

to leading order of $k = \omega\sqrt{\varepsilon_0\mu_0}$. Due to that a number of apparent leading terms vanish in the small k expansion of P_r we observe that the radiated energy couples electric and magnetic sources to lowest order in k.

IV. Q for electrically small volumes

The antenna Q are defined as $Q = \max(Q_E, Q_M)$ where

$$Q_E = \frac{2\omega W_E}{P_r},\ Q_M = \frac{2\omega W_M}{P_r}. \quad (16)$$

the additional factor 2 included in the definition of Q_M Q_M is to simplify the comparison with the antenna Q. We note that to obtain as low Q as possible, we can consider this as a minimization problem with respect to the four sources: $\boldsymbol{J}_{e,m}^{(0,1)}$.

An interesting benchmark of these generalized stored energies is the antenna Q when the volume is electrically small. Already from the form of (15) and (11)-(12) we see that $Q \sim k^{-3}$ to lowest order independent of the shape of the support of the current density sources. This result agree with the special cases of Chu [15] and [2, 7, 17].

A. Q for electric dipole and loop-currents on a sphere

To explicitly determine the constant in the expression for the leading order Q, we consider a spherical shell with a current of dipole-type, *i.e.*, $\boldsymbol{J}_e^{(1)} = J_1\hat{\boldsymbol{\theta}}\delta(r-a)\sin\theta$. Analytical integration yields

$$P_r = \frac{k^4\eta_0}{12\pi}\Big|\int_V \boldsymbol{J}_e^{(1)}\,dV\Big|^2 = |J_1|^2\frac{16\pi(ka)^4\eta_0}{27}, \quad (17)$$

$$W_E = \frac{1}{16\pi\varepsilon_0}\int_V\int_V \frac{\rho_{e1}^{(1)}\rho_{e2}^{(1)*}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,dV_1\,dV_2 = |J_1|^2\frac{4\pi a\mu_0}{9} \quad (18)$$

and $W_M = 0$. We find to leading order:

$$Q = \frac{3}{2(ka)^3}. \quad (19)$$

Assume that the current density is of the loop-type: $\boldsymbol{J}_e = J_0\hat{\boldsymbol{\varphi}}\sin\theta\delta(r-a)$. We insert this expression into the above stored energy and radiated power to find

$$P_r = \frac{k^4}{12\pi\eta_0}\Big|\frac{1}{2}\int_V \boldsymbol{r}\times\boldsymbol{J}_e^{(0)}\,dV\Big|^2 = |J_0|^2\frac{4\pi a^2(ka)^4\eta_0}{27}, \quad (20)$$

$$W_M = \frac{\mu_0}{16\pi}\int_V\int_V \frac{\boldsymbol{J}_{e1}^{(0)}\cdot\boldsymbol{J}_{e2}^{(0)*}}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,dV_1\,dV_2 = |J_0|^2\frac{2\pi a^3\mu_0}{9} \quad (21)$$

Similarly $W_E = 0$ and we find to leading order:

$$Q = \frac{3}{(ka)^3} \quad (22)$$

Both these Q-result are identical to [3, 7, 16, 17] and they are above the Chu-limit since we only use electrical surface currents.

B. Magnetic currents and equivalent currents on a small sphere

If we instead consider purely magnetic sources we find equivalently for a magnetic dipole current and magnetic loop current respective

$$Q_{\text{dipole}} = \frac{3}{2(ka)^3},\quad Q_{\text{loop}} = \frac{3}{(ka)^3}. \quad (23)$$

The results follows directly from the structure of the electric and magnetic stored energies see (9), (10).

It is clear from the structure of P_r and the energies that we can archive lower Q for combined electric and magnetic equivalent surface currents. That is if we combine an electric dipole with a magnetic loop surface current density on the sphere we find that $Q = 1/(ka)^3$ which is the leading order term of Chu [15]. Similarly for the dual case.

Using all four different kinds of equivalent surface current densities: Electric and magnetic dipole together with electric and magnetic loop we find that $Q = Q_E = Q_M = 1/(2(ka)^3)$, as the leading order term, see also [2, 3, 25–27].

V. Conclusion

We have extended the expressions for the stored energies [17] and radiated power to the case of electric and magnetic currents. The new expressions are a quadratic form in the electric and magnetic currents that can be directly used in optimization [11, 12]. The antenna Q is derived in the electrically small case. The electrically small case for the radiated power is, as expected, expressed in terms of the dipole moments. The results agree with known results for spherical structures.

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