A Well-Conditioned Algebraic System for Scattering from Eccentrically Layered Circular Cylinders

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***Abstract*—The regularization of the well-known analytical formulation of the monochromatic electromagnetic wave scattering problem from an eccentrically layered circular cylinder system is presented. It is the improvement and extension of the work done for scattering from two perfectly conducting circular cylinders. Numerical results show that it is numerically much safer to solve the obtained infinite algebraic system at a lower truncation number also by ensuring the reliability of the solution.**

I. Introduction

The layered eccentric circular cylinder (LECC) system is a model that can be useful in a great variety of engineering applications including, bio-electromagnetic models for human lower leg [1] and investigation of cylindrical phase shifting screens as in [2]. The incomplete mode matching (IMM) technique [1] is a versatile approach for obtaining the analytical formulation of this problem. However, IMM's results are open for improvement using the idea in [2] given for the electromagnetic wave scattering problem concerning the two perfectly conducting circular cylinders. The reliability of the infinite series representations of the fields that the IMM obtains after truncation is verified with physical requirements such as the reciprocity of the fields and the continuity of near fields. The regularization of the IMM provides a perspective to understand why the above mentioned verifications are crucial for IMM. The LECC geometry in Fig.1 is used to demonstrate these issues. All formulations will be for the TM mode since the main focus here does not differ for the corresponding IMM-TE mode formulations. The numerical results for verification of the implementation and casting main features of the regularization will be given.

II. Formulation

In Fig. 1, O_m stands for the position vectors of the centres of polar coordinate systems (ρ_m, φ_m) for outer (m=1) and inner (m=2) eccentric circular boundaries (ECB), respectively. They separate the x-y plane into the regions indexed by j=0,1,2, i.e. from the outermost to the innermost. The time dependence of the monochromatic electromagnetic waves is $e^{-i\omega t}$ and the dielectric permittivity ε_j and magnetic permeability μ_j are complex valued quantities in general. All the magnetic field components are expressible via E_z in all regions, which are the unique components of the E-fields for the TM modes. Total E_z in all regions consist of incoming and outgoing components with the index χ equal to "0" and "∞" respectively:

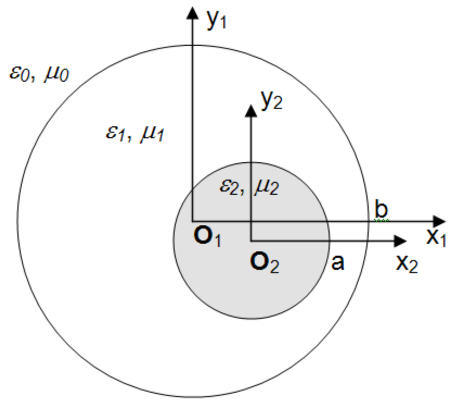


Fig. 1. Geometry of the two layered eccentric circular cylinder system.

$$E_z(\rho_m,\varphi_m)=E_z^{(j,0)}(\rho_m,\varphi_m)+E_z^{(j,\infty)}(\rho_m,\varphi_m)$$
$$E_z^{(j,\chi)}(\rho_m,\varphi_m)=\sum_{n=-\infty}^{\infty} C_n^{(j,\chi)}Z_n^{(\chi)}(k_m\rho_m)e^{in\varphi_m} \qquad (1)$$

Here, $k_j=\omega(\varepsilon_j\mu_j)^{1/2}$, $Z_n^{(\chi)}(t)$ is Bessel function $J_n(t)$ for χ=0 and Hankel function $H_n^{(1)}(t)$ for $\chi=\infty$, and $C_n^{(j,\chi)}$ are the coefficients that are sought for in region j to satisfy the boundary condition i.e. the continuity of the tangential fields E_z and H_φ on the ECBs. Only $C_n^{(0,0)}$ is known for the incident field (see [3] for corresponding expressions for a plane wave or a line source). Moreover we assume that there is no source in region j=2 thus $E_z^{(2,\infty)}\equiv C_n^{(2,\infty)}\equiv 0$. Eccentricity vector is defined in polar coordinates as $\mathbf{d}=(d,\theta)=O_2-O_1$. To express the fields $\chi=0$ and $\chi=\infty$ equivalently to the IMM [1], they are represented by the polar coordinates of the boundary centred to O_m with understanding that it scatters the fields inwards and outwards that boundary. Then after manipulations, the boundary condition stated above results into the following functions that are familiar from analysis of coaxial circular cylindrical layers [6]:

$$
\begin{aligned}
&P_n^{(j,l)}(\rho)=\beta_j H_n^{(1)}(k_l\rho)J'_n(k_j\rho)-\beta_l H_n^{(1)\prime}(k_l\rho)J_n(k_j\rho),\\
&Q_n^{(j,l)}(\rho)=\beta_j J_n(k_l\rho)J'_n(k_j\rho)-\beta_l J'_n(k_l\rho)J_n(k_j\rho),\\
&T_n^{(j,l)}(\rho)=\beta_j H_n^{(1)}(k_l\rho)H_n^{(1)\prime}(k_j\rho)-\beta_l H_n^{(1)\prime}(k_l\rho)H_n^{(1)}(k_j\rho),\\
&W_n^{(j)}(\rho)=-P_n^{(j,j)}(\rho)=2i\beta_j/\pi k_j\rho,\ \beta_j=k_j/\mu_j,\\
&\begin{bmatrix}F_{n,s}^{\infty}(\rho)\\ F_{n,s}^{0}(\rho)\end{bmatrix}=F_s(\rho)J_{\begin{bmatrix}n-s\\ s-n\end{bmatrix}}(k_1 d)e^{i(s-n)\theta},\ \begin{pmatrix}n\\ s\end{pmatrix}=\begin{pmatrix}0,\pm1,\pm2,\dots\\ 0,\pm1,\pm2,\dots\end{pmatrix}
\end{aligned}
\tag{2}
$$

Here ‘ stands for derivative with respect to argument. The last functions in (2) with double indices are results of Graf's addition theorem for Bessel functions [1,3,5] to form the interaction term between the ECBs at the non-zero neighbor diagonal blocks of the 4x4 block infinite matrix A in (3). Single index contributions on diagonal blocks of matrix A are also diagonal entries. Unknown coefficient vector column x is formed by appending 4 vector columns of the unknown coefficients sequentially and vector column b is formed by contributions from the incident field coefficients and with the functions defined via (2). We write the values of the radii $\rho_1=b$ and, $\rho_2=a$ to get to (3).

$$
\underbrace{\begin{bmatrix}
\left[P_n^{(1,0)}(b)\right] & \left[W_{n,s}^{(1)\infty}(b)\right] & [0] & [0]\\
[0] & \left[P_n^{(2,1)}(a)\right] & \left[Q_{n,s}^{(2,1)0}(a)\right] & [0]\\
[0] & \left[T_{n,s}^{(1,0)\infty}(\rho)\right] & \left[P_n^{(1,0)}(b)\right] & [0]\\
[0] & [0] & \left[W_{n,s}^{(1)0}(a)\right] & \left[P_n^{(2,1)}(a)\right]
\end{bmatrix}}_{A}
\underbrace{\begin{bmatrix}C_n^{(0,\infty)}\\ C_n^{(1,\infty)}\\ C_n^{(1,0)}\\ C_n^{(2,0)}\end{bmatrix}}_{x}
=
\underbrace{\begin{bmatrix}C_n^{(0,0)}Q_n^{(0,1)}(b)\\ 0\\ C_n^{(0,0)}W_n^{(0)}(b)\\ 0\end{bmatrix}}_{b}
\tag{3}
$$

Graf's addition theorem is used to express outgoing and incoming waves in region j=1 to meet outer and inner boundaries respectively to equate the Fourier series written according to the corresponding polar coordinate systems of the boundaries.

III. REGULARIZATION

Asymptotic behaviors of the functions $Z_n^{(\chi)}(t)$ in (1) as $n\to\infty$ as well as their derivatives seen in (2) (see [5]), clarify that the entries to matrix A in (3) possess rapidly increasing or decreasing structures as $n,s\to\infty$. In [4], it was recognised that scaling x, in (3), with functions defined on the corresponding boundaries initiates a regularization. They are order-wise asymptotically reciprocal (OWAR) to $Z_n^{(\chi)}(t)$ in (1) and set in operator **R** in (4) for LECC. We suggest here use of the $H_n^{(1)}(t)$ or its reciprocal in **R** instead of $J_n(t)$ (as was done in [4]) or $Y_n(t)$ which also may provide OWAR scaling. This maintains the equivalence to the initial posing of the problem for real valued t since zeros of $H_n^{(1)}(t)$ are not real and therefore the entries of **R** are neither 0 nor ∞ (for complex t attention is still required). Thus the regularizer pair (**L**,**R**) with,

$$
\mathbf{L}=\Lambda I,\mathbf{R}=\Omega\mathbf{I},\ \mathbf{I}: Unit\ Matrix\ Operator,
$$

$$
\Lambda=\left[\left(\frac{P_n^{(1,0)}(b)}{H_n^{(1)}(k_0 b)}\right)^{-1}\ \left(\frac{P_n^{(2,1)}(a)}{H_n^{(1)}(k_1 a)}\right)^{-1}\ \frac{\left(P_n^{(1,0)}(b)\right)^{-1}}{H_n^{(1)}(k_1 b)}\ \frac{\left(P_n^{(2,1)}(a)\right)^{-1}}{H_n^{(1)}(k_2 a)}\right]
\tag{4}
$$

$$
\Omega=\left[\left(H_n^{(1)}(k_0 b)\right)^{-1}\ \left(H_n^{(1)}(k_1 a)\right)^{-1}\ H_n^{(1)}(k_1 b)\ \ H_n^{(1)}(k_2 a)\right]
$$

$$
\mathbf{K}=\begin{bmatrix}0 & [K_I] & 0 & 0\\ 0 & 0 & [K_{II}] & 0\\ 0 & [K_{III}] & 0 & 0\\ 0 & 0 & [K_{IV}] & 0\end{bmatrix}
$$

transforms the infinite algebraic system (IAS) of the first kind **A**x=b to an IAS of the second kind (**I**+**K**)y=g where y,g∈ l_2, as (**I**+**K**)=**LAR**, y=$\mathbf{R}^{-1}$x and g=**L**b. Here I is the unit operator. Letting $\Lambda_{I,II,III,IV}$ be some real valued constants of asymptotic analysis, we find out that, the inequalities for upper bounds of the non-zero entries of **K**,

$$
\begin{aligned}
&\left|k_{ns}^{(1)}\right|<\Lambda_I *\left[(|n|+|s|)!/(|n|!|s|!)\right](d/b)^{|s|}(a/b)^{|n|}\\
&\left|k_{ns}^{(2)}\right|<\Lambda_{II} *\left[(|n|+|s-1|)!/(|n|!|s|!)\right](d/b)^{|s|}(a/b)^{|n|}\\
&\left|k_{ns}^{(3)}\right|<\Lambda_{III} *\left[(|n+1|+|s|)!/(|n|!|s|!)\right](d/b)^{|s|}(a/b)^{|n|}\\
&\left|k_{ns}^{(4)}\right|<\Lambda_{IV} *\left[(|n|+|s|)!/(|n|!|s|!)\right](d/b)^{|s|}(a/b)^{|n|}
\end{aligned}
\tag{5}
$$

prove that K is a compact operator in l_2 as long as d<b-a, a<b and d<b, which are naturally satisfied. Here, $k_{ns}^{(1,2,3,4)}$ are entries for matrix blocks K_I, K_{II}, K_{III}, K_{IV} respectively in (4) and all other blocks are sparse.

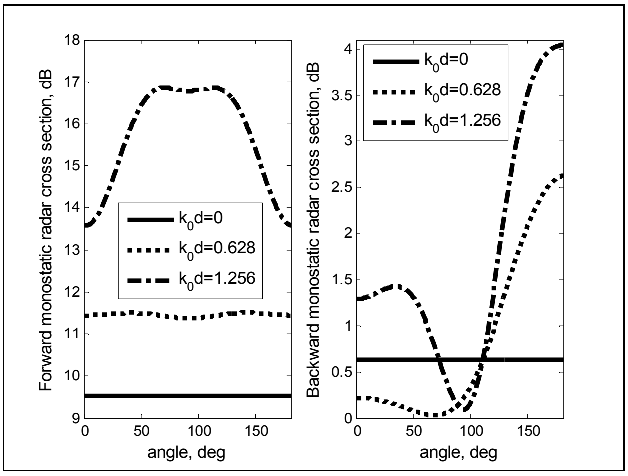
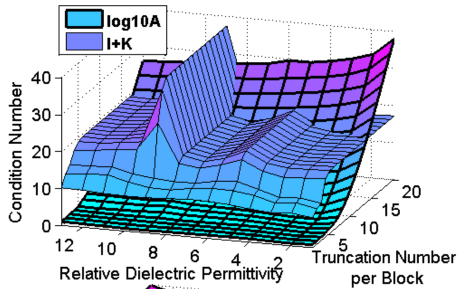


Fig. 2 Forward and backward monostatic radar cross-sections of LECC. θ=0, $k_0b=4$, $k_0a=2$, $\varepsilon_{r1}=2$, $\varepsilon_{r2}=4$, ε_r: Relative dielectric permittivity.

IV. NUMERICAL RESULTS

In Fig.2 the truncation number per block is 6 and the Fig.5 and Fig.6 of [3] are repeated with perfect agreement. Fig.3 depicts the pre- and post-regularization condition numbers (ν) of the IAS for two variations of the case in Fig.2: (above) increasing ε_{r2} and (below) increasing conductivity σ_2. Impact of the regularization for the considered cases is dramatic since pre-regularization IAS ν is inexpressible in the same plot frame without log10 of their values for a clear plot. All the post-regularization IAS ν values themselves in Fig.3 are uniformly bounded. It means that the new unknown $y \in l_2$ can be obtained numerically with correct significant digits starting from small truncation numbers. Also when ε_{r2} 5 and 9 at Fig.3 above, we see peaks in post-regularization ν that is worth closer attention for finding a resonant regime around those parameters. At Fig.3 below, it is noteworthy that presence of conductivity worsens the ill-conditioning of pre-regularization IAS.



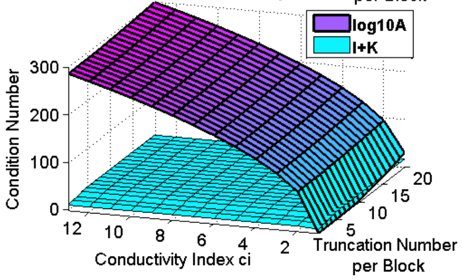


Fig. 3 Condition number ν of IAS for Fig.2 parameters (left) varying ε_{r2}, (right) with $\varepsilon_{r2}=4+i\sigma/(\varepsilon_0\omega)$, varying conductivity and $k_0d=1.256$. ε_0: Free space dielectric permittivity, σ:Conductivity in S/m ((ci-1)x50S/m), ω:angular frequency (2π50MHz).

V. Conclusion

We revived an old regularization approach with its improved new version applying it to another canonical problem used in modelling a variety of engineering applications in the ambit of results with higher precision. The regularized approach provides a mathematically rigorous and stable numerical evaluation process. It alleviates the primary importance of the verification of the algorithm with physical requirements like near field continuity and reciprocity. Since the pre-regularization truncated system is numerically ill-conditioned these physical checks are crucial to evaluate the problem numerically without regularization.

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