On the generalized Jordan's lemma with applications in waveguide theory

Sven Nordebo, Stefan Gustafsson, Andreas Ioannidis, Börje Nilsson, Joachim Toft

School of Computer Science, Physics and Mathematics,
Linnæus University, 351 95 Växjö, Sweden
{sven.nordebo,stefan.h.gustafsson,andreas.ioannidis,borje.nilsson,joachim.toft}@lnu.se

***Abstract*—This paper presents two variants of a generalized Jordan's lemma with applications in waveguide theory. As a main application is considered an asymptotic analysis for open waveguide structures with circular geometry. In particular, the generalized Jordan's lemma can be used to justify that field components can be calculated as the sum of discrete and non-discrete modes, *i.e.*, as the sum of residues and an integral along the branch-cut defined by the transversal wavenumber of the exterior domain. An explicit example regarding the axial symmetric TM$_0$ modes of a single core transmission line, wire, or optical fibre is included to demonstrate the associated asymptotic behavior for a typical open waveguide structure.**

I. Introduction

The purpose of this paper is to provide a generalization of Jordan's lemma with applications in waveguide theory. Two variants of a generalized lemma are presented which can be used as a basis for asymptotic analysis and a study of non-discrete (radiating) modes of complicated open waveguide structures with circular geometry, such as HVDC power cables [1], [2], [3] or optical fibers [4], [5], [6]. In particular, asymptotic approximation methods for Fourier integrals including rigorous error bounds [3], [7] can be used in order to evaluate the contribution from the branch-cut integral in comparison to the residues [3].

The present description is based primarily on observations made with respect to the axial symmetric fields of a multi-layered coaxial cable [8]. In [8] is given a layer-recursive description of the dispersion function for axial symmetric Transverse Magnetic (TM) fields, and which is well-suited for asymptotic analysis. The recursion is based on two well-behaved (meromorphic) sub-determinants which are defined by a perfectly electric conducting (PEC) and a perfectly magnetic conducting (PMC) termination, respectively. In particular, the analysis in [8] shows that for a multi-layered open waveguide structure, the dispersion function contains only one branch-point, and which is related to the exterior domain.

Another crucial observation is that the basic condition for the standard Jordan's lemma [9], [10] is not generally satisfied with open waveguide structures. Hence, the associated Fourier transform does not vanish on the appropriate semicircle in the complex plane, as the radius of the semicircle tends to infinity.

The literature on generalized versions of Jordan's lemma is not vast, see *e.g.*, [11], [12]. In [11] on p. 245, a generalized Jordan's lemma is defined concerning a rotation of the associated semicircle. This lemma, however, assumes that the transform has the same asymptotic behavior at infinity as with the standard Jordan's lemma. In [12], an extended Jordan's lemma is presented where the Fourier transform is assumed to have a sub-exponential growth on the whole semicircle in the upper half of the complex plane. However, the extended lemma as it is stated in [12] does not seem to be rigorously established. For this reason, two different variants of a generalized Jordan's lemma are given here. The first variant can be regarded as a modification of the extended lemma presented in [12]. The second variant is a true generalization of the standard Jordan's lemma in the sense that the standard lemma becomes a special case of the generalized lemma. Both variants of the generalized lemma appear to be adequate for open waveguide structures, with the purpose of justifying that field components can be calculated as the sum of discrete and non-discrete modes.

II. Field components in circular geometry

The vector components of the electromagnetic field in a multi-layered waveguide with circular geometry can generally be written in terms of an inverse Fourier integral as follows

$$\varphi_m(\rho,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\alpha)}{G(\alpha)}\mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}\alpha=\int_{-\infty}^{\infty}f(\alpha)\mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}\alpha,\quad(1)$$

where ρ and z are the radial and longitudinal cylindrical coordinates, m the azimuthal index and α the Fourier variable. The function $f(\alpha)=F(\alpha)/(2\pi G(\alpha))$ consists of various rational combinations of Bessel functions and Hankel functions of the first and second kind and order m. Here, $G(\alpha)$ is the (transcendental) dispersion function obtained as the determinant of the linear system of equations representing the boundary conditions at hand, and $F(\alpha)$ is determined by the excitation, or sources that are present, see *e.g.*, [1], [8], [13], [14]. It should be noted that the function $f(\alpha)$ in (1) depends also on the radial coordinate ρ.

The arguments of the Bessel and Hankel functions contain the transversal wavenumbers

$$\kappa=\sqrt{k_0^2\mu\epsilon-\alpha^2},\quad(2)$$

for each cylindrical layer, and where μ and ϵ are the corresponding relative permeability and permittivity, respectively. Here $k_0=\omega/\mathrm{c}_0$ is the wavenumber in vacuum, c_0 the speed of light in vacuum, and ω the angular frequency. The relative permeability and permittivity in each layer are generally complex valued for lossy materials. The square root $\kappa = \sqrt{w}$ in (2) is defined such that $0 < \arg w \leq 2\pi$ and $0 < \arg\kappa \leq \pi$ and hence $\operatorname{Im}\kappa \geq 0$, see *e.g.*, [15].

For a multi-layered open waveguide structure with circular geometry, the function $f(\alpha)$ in (1) exhibits only one branch-point, and which is related to the exterior domain, see [8]. This branch-point is denoted $\alpha_{\rm c}$ and is given by $\alpha_{\rm c} = k_0\sqrt{\mu_{\rm e}\epsilon_{\rm e}}$, where $\mu_{\rm e}$ and $\epsilon_{\rm e}$ are the relative permeability and permittivity of the exterior domain, respectively. The corresponding branch-cut for the square root $\kappa = \sqrt{k_0^2\mu_{\rm e}\epsilon_{\rm e} - \alpha^2}$ is denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$ in the α-plane, and $\mathcal{D}_1$ and $\mathcal{D}_2$ in the κ-plane, as depicted in Fig. 1.

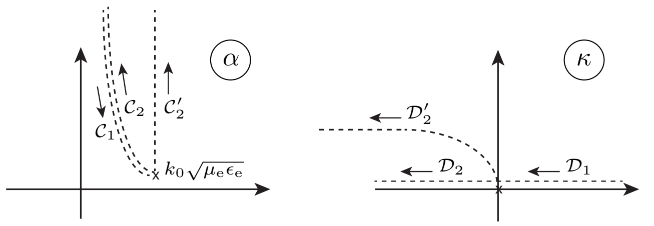

Fig. 1. Integration contours for an open waveguide structure with losses. Here, $\mathcal{C}_1$ and $\mathcal{C}_2$ are the original contours associated with the branch-cut at $\alpha_{\rm c} = k_0\sqrt{\mu_{\rm e}\epsilon_{\rm e}}$, and $\mathcal{C}_2'$ the deformed contour. The corresponding contours in the κ-plane are denoted $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_2'$ where $\kappa = \sqrt{k_0^2\mu_{\rm e}\epsilon_{\rm e} - \alpha^2}$.

Note that the transversal wavenumbers in (2) will tend to infinity as

$$\kappa = \sqrt{k_0^2\mu\epsilon - \alpha^2} \sim \pm{\rm i}\alpha, \tag{3}$$

as $\alpha \to \infty$. Since the asymptotic behavior of the Hankel functions are given by ${\rm H}_m^{(1)}(\kappa\rho) \sim \sqrt{2/(\pi\kappa\rho)}{\rm e}^{{\rm i}(\kappa\rho - m\pi/2 - \pi/4)}$ and ${\rm H}_m^{(2)}(\kappa\rho) \sim \sqrt{2/(\pi\kappa\rho)}{\rm e}^{-{\rm i}(\kappa\rho - m\pi/2 - \pi/4)}$ for large arguments [16], the asymptotic behavior of the function $f(\alpha)$ is generally of a form similar to

$$f(\alpha) \sim M\alpha^p{\rm e}^{{\rm i}\kappa a}, \tag{4}$$

as $\alpha \to \infty$, and where M is a constant, $p \geq 0$ an integer and a a positive real number. It follows that

$$f(\alpha) \to 0, \tag{5}$$

as $R \to \infty$, $\alpha = R{\rm e}^{{\rm i}\theta}$, $0 \leq \theta \leq \pi$ and $\theta \neq \pi/2$. However, for $\theta = \pi/2$, it is seen that

$$f(\alpha) \sim M\alpha^p{\rm e}^{\pm{\rm i}Ra}, \tag{6}$$

as $R \to \infty$ and $\alpha = {\rm i}R$. Hence, the standard Jordan's lemma [9], [10] can not be used here. However, a generalized Jordan's lemma is readily adapted to the asymptotics in (4), and which can be used to justify that the integral in (1) can be closed in the upper half of the complex plane where $\operatorname{Im}\alpha \geq 0$.

III. The generalized Jordan's lemma

The following two variants of a generalized Jordan's lemma are useful in waveguide theory, in particular with open waveguide structures. In both variants, it is assumed that $f(\alpha)$ is an analytic function in the upper half of the complex plane where $\operatorname{Im}\alpha \geq 0$, except for a branch-cut and a countable set of isolated singularities (poles). Further, the symbol $o(\cdot)$ denotes order less than $(\cdot)$, see [7]. The first lemma below is a modification of the extended Jordan's lemma presented in [12], and which has an elementary proof as outlined below.

***Lemma* III.1.** (Generalized Jordan's lemma I)
Let $\mathcal{C}_R$ be the upper semicircle defined by $\alpha = R{\rm e}^{{\rm i}\theta}$ with radius $R > 0$ and angle $0 \leq \theta \leq \pi$. Let $z > 0$, and assume that

$$\alpha f(\alpha){\rm e}^{{\rm i}\alpha z} = o(1) \text{ as } |\alpha| \to \infty \text{ and } \alpha \in \mathcal{C}_R, \tag{7}$$

and where the convergence is uniform. Then the following identity is valid

$$\lim_{R\to\infty}\int_{\mathcal{C}_R} f(\alpha){\rm e}^{{\rm i}\alpha z}\,{\rm d}\alpha = 0. \tag{8}$$

Proof: Let $\epsilon > 0$ be arbitrary and R sufficiently large so that $|\alpha f(\alpha){\rm e}^{{\rm i}\alpha z}| \leq \epsilon$ for $\alpha \in \mathcal{C}_R$. Now,

$$\begin{aligned}\left|\int_{\mathcal{C}_R} f(\alpha){\rm e}^{{\rm i}\alpha z}\,{\rm d}\alpha\right| &\leq \int_{\mathcal{C}_R} |f(\alpha){\rm e}^{{\rm i}\alpha z}||\,{\rm d}\alpha| \\ &\leq \pi R\sup_{\mathcal{C}_R}|f(\alpha){\rm e}^{{\rm i}\alpha z}| = \pi\sup_{\mathcal{C}_R}|\alpha f(\alpha){\rm e}^{{\rm i}\alpha z}| \leq \pi\epsilon. \end{aligned}\tag{9}$$

Since the right-hand side of (9) above can be made arbitrarily small, this shows that (8) is valid. ∎

Even though the lemma III.1 is sufficient for our purposes, it does not cover all the cases that are covered by the standard Jordan's lemma [9], [10]. In particular, the lemma III.1 requires that $\alpha f(\alpha) = o(1)$ on the real axis where $\operatorname{Im}\alpha = 0$, whereas the standard lemma only requires that $f(\alpha) = o(1)$. For this reason, another variant of the generalized Jordan's lemma can be formulated which includes the standard lemma. This is achieved by dividing the upper semicircle into two subsectors, and by exploiting the inequality $\sin\theta \geq 2\theta/\pi$ which is valid for $0 \leq \theta \leq \pi/2$.

***Lemma* III.2.** (Generalized Jordan's lemma II)
Let β be an angle in the interval $0 \leq \beta < \pi/2$. Let $\mathcal{C}_{R_1}$ be the contour defined by $\alpha = R{\rm e}^{{\rm i}\theta}$ where $0 \leq \theta \leq \pi/2 - \beta$ and $\pi/2 + \beta \leq \theta \leq \pi$, and let $\mathcal{C}_{R_2}$ be the contour defined by $\alpha = R{\rm e}^{{\rm i}\theta}$, $\pi/2 - \beta < \theta < \pi/2 + \beta$, and where $R > 0$ is the radius of the corresponding sectors, see Fig. 2. The contour $\mathcal{C}_{R_2}$ is void when $\beta = 0$. Suppose that the function $f(\alpha)$ has the following asymptotic behavior for large α.

1) *The function $f(\alpha)$ converges uniformly to zero on $\mathcal{C}_{R_1}$, i.e.,*

$$f(\alpha) = o(1) \text{ as } |\alpha| \to \infty \text{ and } \alpha \in \mathcal{C}_{R_1}. \tag{10}$$

2) *The function $f(\alpha)$ has sub-exponential growth on $\mathcal{C}_{R_2}$, i.e.,*

$$f(\alpha) = o({\rm e}^{a|\alpha|}) \text{ as } |\alpha| \to \infty \text{ and } \alpha \in \mathcal{C}_{R_2}, \tag{11}$$

where $a>0$ is an arbitrary constant and the convergence is uniform.

The following identity is valid under the assumptions 1 and 2 above

$$\lim_{R\to\infty}\int_{\mathcal{C}_{R_1}\cup\mathcal{C}_{R_2}} f(\alpha)\mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}\alpha = 0, \tag{12}$$

where $z>0$.

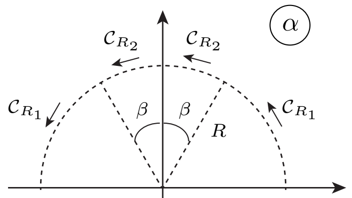

Fig. 2. Integration contours with the generalized Jordan's lemma.

Proof: The integral over $\mathcal{C}_{R_1}$ is evaluated as follows. Let $\epsilon>0$ be arbitrary and R sufficiently large so that $|f(\alpha)|\le\epsilon$ for $\alpha\in\mathcal{C}_{R_1}$. Now, with $\alpha=R\mathrm{e}^{\mathrm{i}\theta}$

$$\begin{aligned}\left|\int_{\mathcal{C}_{R_1}} f(\alpha)\mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}\alpha\right| &= \left|\int_{\mathcal{C}_{R_1}} f(\alpha)\mathrm{e}^{\mathrm{i}Rz\mathrm{e}^{\mathrm{i}\theta}}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\,\mathrm{d}\theta\right|\\
&\le \int_{\mathcal{C}_{R_1}} |f(\alpha)|\mathrm{e}^{-Rz\sin\theta}R\,\mathrm{d}\theta \le \epsilon R\int_{\mathcal{C}_{R_1}}\mathrm{e}^{-Rz\sin\theta}\,\mathrm{d}\theta\\
&= 2\epsilon R\int_0^{\pi/2-\beta}\mathrm{e}^{-Rz\sin\theta}\,\mathrm{d}\theta \le 2\epsilon R\int_0^{\pi/2-\beta}\mathrm{e}^{-Rz2\theta/\pi}\,\mathrm{d}\theta\\
&= \frac{\pi\epsilon}{z}\left(1-\mathrm{e}^{-Rz(1-2\beta/\pi)}\right),\end{aligned} \tag{13}$$

where $\sin\theta>2\theta/\pi$ for $0\le\theta\le\pi/2-\beta$ has been used. Since the right-hand side of (13) above can be made arbitrarily small, this shows that

$$\lim_{R\to\infty}\int_{\mathcal{C}_{R_1}} f(\alpha)\mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}\alpha = 0, \tag{14}$$

where $z>0$.

The integral over $\mathcal{C}_{R_2}$ is now evaluated as follows. Let $z>0$ be arbitrary, but fixed. Let $\epsilon>0$ be arbitrary and R sufficiently large so that $|f(\alpha)|\le\epsilon\mathrm{e}^{aR}$ for $\alpha\in\mathcal{C}_{R_2}$. Again, with $\alpha=R\mathrm{e}^{\mathrm{i}\theta}$

$$\begin{aligned}\left|\int_{\mathcal{C}_{R_2}} f(\alpha)\mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}\alpha\right| &= \left|\int_{\mathcal{C}_{R_2}} f(\alpha)\mathrm{e}^{\mathrm{i}Rz\mathrm{e}^{\mathrm{i}\theta}}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\,\mathrm{d}\theta\right|\\
&\le \int_{\mathcal{C}_{R_2}} |f(\alpha)|\mathrm{e}^{-Rz\sin\theta}R\,\mathrm{d}\theta \le \epsilon R\mathrm{e}^{aR}\int_{\pi/2-\beta}^{\pi/2+\beta}\mathrm{e}^{-Rz\sin\theta}\,\mathrm{d}\theta\\
&= 2\epsilon R\mathrm{e}^{aR}\int_{\pi/2-\beta}^{\pi/2}\mathrm{e}^{-Rz\sin\theta}\,\mathrm{d}\theta\\
&\le 2\epsilon R\mathrm{e}^{aR}\int_{\pi/2-\beta}^{\pi/2}\mathrm{e}^{-Rz2\theta/\pi}\,\mathrm{d}\theta\\
&= \frac{\pi\epsilon}{z}\mathrm{e}^{aR}\left(\mathrm{e}^{-Rz(1-2\beta/\pi)}-\mathrm{e}^{-Rz}\right)\\
&= \frac{\pi\epsilon}{z}\left(\mathrm{e}^{-R(z(1-2\beta/\pi)-a)}-\mathrm{e}^{-R(z-a)}\right),\end{aligned} \tag{15}$$

where $\sin\theta>2\theta/\pi$ for $\pi/2-\beta\le\theta\le\pi/2$ has been used. Note that $0<\beta<\pi/2$ so that $0<1-2\beta/\pi<1$. Hence, a can be chosen so that

$$a<z(1-2\beta/\pi)<z. \tag{16}$$

Since the right-hand side of (15) above can be made arbitrarily small, this shows that

$$\lim_{R\to\infty}\int_{\mathcal{C}_{R_2}} f(\alpha)\mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}\alpha = 0, \tag{17}$$

where $z>0$. Note that when $\beta=0$ the contour $\mathcal{C}_{R_2}$ above is void and the contour $\mathcal{C}_{R_1}$ consists of the whole semicircle for $0\le\theta\le\pi$. This situation corresponds to the standard version of the Jordan's lemma [9], [10]. ■

It should be noted that even though the two variants of the generalized Jordan's lemma III.1 and III.2 above are similar, they are not equivalent and one can not say that one is more general than the other.

IV. Asymptotic analysis

Based on the generalized Jordan's lemma III.1 or III.2 together with the asymptotic behavior in (5) and (6), the integral in (1) can be written as

$$\varphi_m(\rho,z) = \sum_j \frac{\mathrm{i}F(\alpha_j)}{G'(\alpha_j)}\mathrm{e}^{\mathrm{i}\alpha_j z} + \int_{\mathcal{C}_2} q(\alpha)\mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}\alpha, \tag{18}$$

where α_j are the poles, the function $q(\alpha)$ is given by

$$q(\alpha) = f(\alpha)|_{\mathcal{C}_2} - f(\alpha)|_{\mathcal{C}_1}, \tag{19}$$

and $\mathcal{C}_1$ and $\mathcal{C}_2$ are the branch-cut contours depicted in Fig. 1.

The function $q(\alpha)$ is analytic by continuation except for the branch-point α_{c}, and the contour $\mathcal{C}_2$ can therefore be deformed to the vertical contour $\mathcal{C}_2'$ depicted in Fig. 1. Note that the vertical contour $\mathcal{C}_2'$ corresponds to the contour $\mathcal{D}_2'$ in the κ-plane where κ approaches the asymptot $\operatorname{Im}\kappa = \operatorname{Re}k_0\sqrt{\mu_{\mathrm{e}}\epsilon_{\mathrm{e}}}$ as $\alpha\to\infty$. Hence, factors of the type $\mathrm{e}^{\mathrm{i}\kappa a}$ will be oscillating, but do not decay exponentially on $\mathcal{C}_2'$. Assume that the function $q(\alpha)$ and possibly some of its derivatives $q^{(n)}(\alpha)$ for $n=0,1,\ldots,N$, are defined and continuous at the branch-point α_{c}. It follows then from (4) that $q(\alpha)$ and $q^{(n)}(\alpha)$ are upper bounded on the contour $\mathcal{C}_2'$ as

$$|q^{(n)}(\alpha)| \le M^{(n)}(1+|\alpha-\alpha_{\mathrm{c}}|^{p_n}), \tag{20}$$

where $M^{(n)}$ is a constant, p_n an integer and $n=0,1,\ldots,N$. One can also assume that there will be a logarithmic singularity in one of the first derivatives of $q(\alpha)$, and which can be expressed as

$$q^{(N+1)}(\alpha) = A\ln(-\mathrm{i}(\alpha-\alpha_{\mathrm{c}})) + r^{(N+1)}(\alpha), \tag{21}$$

where A is a constant and $r^{(N+1)}(\alpha)$ a function which is continuous at α_{c} and upper bounded on the contour $\mathcal{C}_2'$ as

$$|r^{(N+1)}(\alpha)| \le M^{(N+1)}(1+|\alpha-\alpha_{\mathrm{c}}|^{p_{N+1}}), \tag{22}$$

where $M^{(N+1)}$ is a constant and p_{N+1} an integer.

Explicit expressions for the derivatives $q^{(n)}(\alpha_c)$ and the constant A can be obtained by a detailed study of the function $q(\alpha)$ defined in (19). Here, it is useful to employ the identity $\kappa|_{\mathcal{C}_1} = -\kappa|_{\mathcal{C}_2}$ and $\mathrm{H}_m^{(1)}(\kappa\rho)|_{\mathcal{C}_1} = \mathrm{H}_m^{(1)}(-\kappa\rho)|_{\mathcal{C}_2} = -(-1)^m\mathrm{H}_m^{(2)}(\kappa\rho)|_{\mathcal{C}_2}$, to account for the discontinuity over the branch-cut, and where $\kappa = \sqrt{k_0^2\mu_e\epsilon_e - \alpha^2}$. An asymptotic study of the branch-cut integral in (18) can now be pursued by integration by parts as described in [3], [7].

V. Example

As an explicit example of an open waveguide structure is considered a single core transmission line, wire, or optical fibre, with radius a. The relative permeability and permittivity of the inner and the outer exterior domain are denoted by μ_1, ϵ_1, μ_2 and ϵ_2, respectively, see Fig. 3.

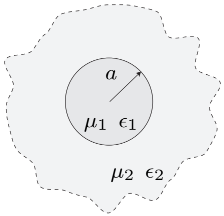

Fig. 3. Circular waveguide, wire, or optical fibre, with an exterior domain.

A vector point-source $\boldsymbol{J} = P_\rho\hat{\boldsymbol{\rho}}\frac{1}{\rho}\delta(\rho-\rho')\delta(\phi)\delta(z)$ is considered as a source for the axial symmetric TM_0 modes. The dipole source is placed close to the inner radius at $\rho' = a-$. The electric field component $E_z(\rho,\phi,z)$ is given by

$$E_z(\rho,\phi,z) = \int_{-\infty}^{\infty} f(\alpha)\mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}\alpha, \tag{23}$$

where a closed form solution for $\rho < a$ is given by

$$f(\alpha) = \frac{1}{2\pi}P_\rho\frac{\eta_0}{2\pi}\frac{\alpha}{a}\frac{\epsilon_2}{\epsilon_1}\frac{\kappa_1}{k_0} \frac{\mathrm{H}_1^{(1)}(\kappa_2 a)\mathrm{J}_0(\kappa_1\rho)}{\epsilon_1\kappa_2\mathrm{J}_1(\kappa_1 a)\mathrm{H}_0^{(1)}(\kappa_2 a) - \epsilon_2\kappa_1\mathrm{J}_0(\kappa_1 a)\mathrm{H}_1^{(1)}(\kappa_2 a)}, \tag{24}$$

where η_0 is the wave impedance of vacuum and where $\kappa_1 = \sqrt{k_0^2\mu_1\epsilon_1 - \alpha^2}$ and $\kappa_2 = \sqrt{k_0^2\mu_2\epsilon_2 - \alpha^2}$.

By using the asymptotic approximations of the Hankel functions for large arguments, it is found that

$$f(\alpha) \sim \frac{1}{2\pi}P_\rho\frac{\eta_0}{2\pi}\frac{\epsilon_2}{\epsilon_1}\frac{1}{k_0 a} \frac{\alpha\kappa_1\sqrt{\frac{a}{\rho}}\left(\mathrm{e}^{\mathrm{i}\kappa_1(a+\rho)}\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}} + \mathrm{e}^{\mathrm{i}\kappa_1(a-\rho)}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\right)}{\epsilon_1\kappa_2\left(\mathrm{e}^{\mathrm{i}2\kappa_1 a}\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}} + \mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}\right) - \epsilon_2\kappa_1\left(\mathrm{e}^{\mathrm{i}2\kappa_1 a}\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}} + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\right)}, \tag{25}$$

as $\alpha \to \infty$ and $\rho < a$. Note that as $\alpha \to \infty$, $\kappa = \sqrt{k_0^2\mu\epsilon - \alpha^2} \to \pm\mathrm{i}\alpha$ with $\operatorname{Im}\kappa \geq 0$. Hence, it is seen that the asymptotics in (25) satisfies the requirements (7), (10) and (11) of both the generalized Jordan's lemmas III.1 and III.2 (with $0 < \beta < \pi/2$), and the integration path used in (23) can therefore be closed in the upper half of the complex plane where $\operatorname{Im}\alpha > 0$.

VI. Conclusion

Two variants of a generalized Jordan's lemma has been formulated which can be used to establish that field components of an open waveguide structure can be calculated as the sum of residues (discrete modes) and an integral along the branch-cut corresponding to the transversal wavenumber of the exterior domain (non-discrete modes). As an explicit example of an open waveguide structure is considered the axial symmetric Transverse Magnetic (TM) modes of a single core transmission line, wire, or optical fibre.

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