Numerical Examination on Effective Permittivity of Periodic Structure by the FDTD Method

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***Abstract*—The effective permittivity of the one-dimensional periodic structure is examined using the reflectance and transmissivity obtained by the FDTD method. By using the reflectance and transmissivity by the FDTD method, the effective permittivity of the dielectric slab, which has the same reflectance and transmissivity with that of periodic structure, is obtained by using the transcendental equation. Also, the effective permittivity is calculated by Rytov approximation, and the range of application and the validity of the effective permittivity are shown by comparing the Rytov approximation with the approximation obtained by the FDTD method.**

I. Introduction

The scattering problem from the periodic structure is one of the important and basic issues of the electromagnetic theory, and has been investigated from the theoretical and numerical viewpoints [1]–[3].

So far, the techniques to analyze the scattering properties of periodic structures have been proposed. However, since these techniques are using complex calculations and special functions, these are not necessarily handled always easily. For the random media, the effective index is important to examine the scattering properties statistically [4]. Since the periodic structure is generally complex, the structure which replaces the periodic structure with the slab has been proposed in order to approximate the electromagnetic wave scattering problem and various methods to obtain the effective permittivity have been proposed [5]–[7]. The effective permittivity of the periodic layer which consists of two dielectrics has been developed and derived the transcendental equation relating the effective permittivity of the periodic structure [8].

In this paper, the effective permittivity of the one-dimensional periodic structure is examined numerically using the FDTD method [6], [9]. At first, the reflectance and transmissivity from the one-dimensional periodic structure is obtained by the FDTD method. Then, the effective permittivity of the dielectric slab, which gives the same reflectance and transmissivity as those of the periodic structure, is obtained by using the transcendental equation. We compare the effective permittivity of various shapes of the grating, which is obtained by the above procedure, with those by the Rytov approximation [7], and show the validity of the present procedure in order to obtain the effective permittivity of the one-dimensional periodic structure.

II. Formulation

In this section, the procedure to obtain the effective permittivity of one-dimensional periodic structure is expressed. It is assumed that the incident plane wave with incident angle θ_i is polarized along the z-axis, which corresponds to an E-Polarized wave. At first, the reflectance and transmissivity from the periodic structure are calculated by the FDTD method [9]. After that, the periodic structure is replaced with the homogeneous dielectric slab with the effective permittivity. In order to obtain the effective permittivity, the reflectance and transmissivity from the dielectric slab with the thickness d and the refractive index n_2 is calculated. The slab is placed in a background medium with the refractive index n_1. For these situations, the reflectance R_{slab} and transmissivity T_{slab} can be obtained as follows:

$$R_{slab}(n_2) = \frac{4R_m \sin^2\left(\frac{\delta}{2}\right)}{(1-R_m)^2 + 4R_m \sin^2\left(\frac{\delta}{2}\right)}, \tag{1}$$

$$T_{slab}(n_2) = \frac{(1-R_m)^2}{(1-R_m)^2 + 4R_m \sin^2\left(\frac{\delta}{2}\right)}, \tag{2}$$

where R_m is the reflectance at each boundary and is given by

$$R_m = \left[\frac{\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}\right]^2. \tag{3}$$

The phase difference δ is expressed by

$$\delta = \frac{4\pi n_2 d}{\lambda_0}\sqrt{n^2 - \sin^2\theta_i} \tag{4}$$

where λ_0 is the wavelength of the incident wave in vacuum, and $n = n_2/n_1$ is the relative refractive index.

In this paper, the effective permittivity can be obtained by using the following transcendental equations:

$$f(\varepsilon_{eff}) = R_{slab}(\varepsilon_{eff}) - R_{FDTD} = 0, \tag{5}$$

$$f(\varepsilon_{eff}) = T_{slab}(\varepsilon_{eff}) - T_{FDTD} = 0, \tag{6}$$

where R_{FDTD} and T_{FDTD} are the reflectance and transmissivity which are calculated by the FDTD method.

On the other hand, consider one-dimensional periodic structure with rectangular dielectrics with the thickness d and periodically changing refractive index n_{lo} and n_{hi} $(n_{lo} < n_{hi})$. The effective refractive index of this structure can be expressed by the following relationship using Rytov approximation [7]:

$$\sqrt{n_{hi}^2 - n_{eff}^2}\tan\left[\pi\sqrt{n_{hi}^2 - n_{eff}^2}\frac{f\Lambda}{\lambda_0}\right] = -\sqrt{n_{lo}^2 - n_{eff}^2}\tan\left[\pi\sqrt{n_{lo}^2 - n_{eff}^2}\frac{(1-f)\Lambda}{\lambda_0}\right] \quad (7)$$

where Λ is the grating period and f is the volume fraction.

III. Numerical Results

In this section, the effective permittivity of three kinds of the structures such as the rectangular cylinders, circular cylinders, and equilateral triangular cylinders is examined by using the procedure in the previous section. The wavelength is assumed to be 1.55μm, and the angle of the incidence is $\theta_i = 0$. The relative permittivity of the three kinds of the periodic structures is set as $\varepsilon_r = 2.0$. The cell size for the FDTD calculation is $\Delta x = \Delta y = \lambda_0/200$ in the case of the rectangular and circular cylinders. For the equilateral triangular cylinders, the cell size is respectively, $\Delta x = \Delta y = \lambda_0/400$. The period is $\Lambda = 0.5\lambda_0$.

In what follows, the relative effective permittivity properties of three kinds of the structures are examined from the viewpoints of the volume fraction and the normalized frequency. The relative permittivity of the background is assumed to be free space.

A. Case of rectangular cylinders

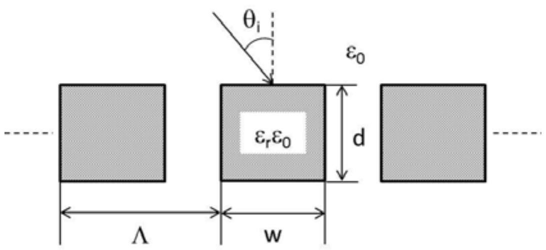

Fig. 1. Periodic structure of rectangular cylinders.

Figure 1 shows the geometry of the reflection from the rectangular cylinder. d is the thickness and w is the width of rectangular cylinder $(d = w)$. The thickness of equivalent dielectric slab is d.

Figure 2 shows the relative effective permittivity for the volume fraction. The symbols ε_{effR} and ε_{effT} mean the relative effective permittivity determined from reflectance and transmissivity of the periodic structure, respectively. "Rytov's approximation" indicates the relative effective permittivity by using Eq. (7). From this figure, it is found that the properties of the relative effective permittivity ε_{effR} and ε_{effT} are good

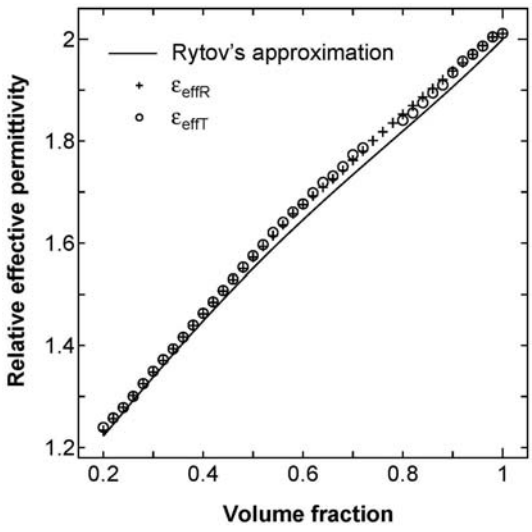

Fig. 2. Relative effective permittivity for rectangular cylinders periodic structure.

agreement with the results by the Rytov's approximation. This is because the derivation of Rytov's approximation is based on reflection from the rectangular dielectric cylinders.

B. Case of circular cylinders

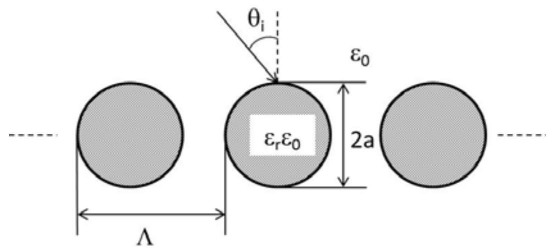

Fig. 3. Periodic structure of circular cylinders.

Figure 3 shows the geometry of the reflection from the circular cylinders with the radius a. The thickness of equivalent dielectric slab is the same as the diameter of the cylinder $2a$.

Figure 4 shows the relative effective permittivity for the volume fraction. From this figure, it is found that the relative effective permittivity ε_{effR} and ε_{effT} can approximate with sufficient accuracy by Rytov's approximation to about 0.35 volume fraction. Also, the Rytov's approximation is not good agreement with the results obtained by the present procedure when the volume fraction is 0.35 or more. This is because Eq. (7) is derived by using the reflection from the rectangular dielectric cylinders, and the difference between the cross sections and shapes of the structures affect the results as the volume fraction increases. In addition to this, Rytov' approximation is satisfied under the condition $\sqrt{\varepsilon_r}d/\lambda_0 \ll 1$ where d is the distance. Around the volume fraction 0.45, the effective permittivity is discontinuous, because the transcendental equation does not have the solution around $\varepsilon_{eff} \sim 2$.

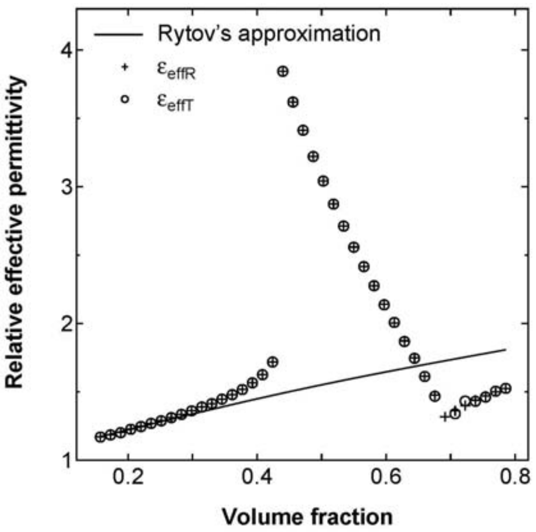

Fig. 4. Relative effective permittivity for circular cylinders for the volume fraction.

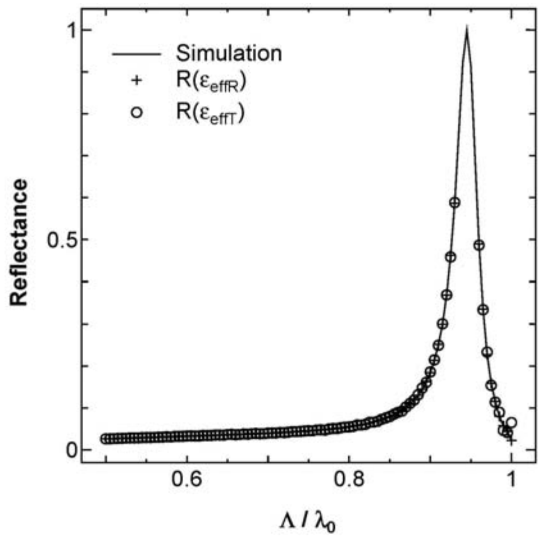

Fig. 6. Reflectance from the periodic structure of the circular cylinders for the normalised frequency Λ/λ_0. for 0.31 of the volume fraction

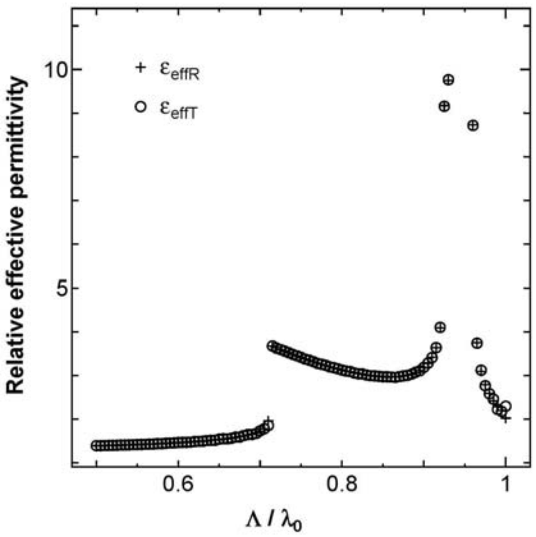

Fig. 5. Relative effective permittivity for circular cylinders for the normalised frequency Λ/λ_0 for 0.31 of the volume fraction.

The relative effective permittivity for the normalized frequency for 0.31 of the volume frequency is shown in Fig. 5. It is found that the discontinuous point exists around $\Lambda/\lambda_0 = 0.71$. This is because the transcendental equation does not have the root and jump to the next point around this frequency. The reason why this phenomenon appears is now under the examination.

Figure 6 shows the reflectance for the normalised frequency Λ/λ_0 for 0.31 of the volume fraction. The normalized radius is $a/\Lambda = 0.2$. The solid line indicates the reflectance obtained by the Method of Moments [10]. It is seen that the reflectance, which is calculated using the effective permittivity, is good agreement with that obtained by MoM by 0.6 or less. In addition to this, the reflectance around $\Lambda/\lambda_0 = 0.71$ changes smoothly although the relative effective permittivity is discontinuous as shown in Fig. 5.

C. Case of equilateral triangular cylinders

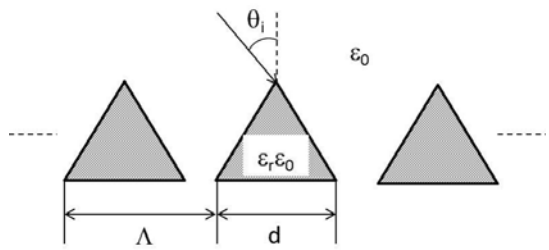

Fig. 7. Periodic structure of equilateral triangular cylinders.

Figure 7 shows the equilateral triangular cylinder with the equal side d. The thickness of equivalent dielectric slab is $\sqrt{3}d/2$.

Figure 8 shows the relative effective permittivity for the volume fraction. From this figure, it is found that the relative effective permittivity ε_{effR} and ε_{effT} can approximate with sufficient accuracy by Rytov's approximation to about 0.25 volume fraction. Also, we can find the discontinuous point appears around 0.35 of the volume fraction as the same as the case of the circular cylinders.

Figure 9 shows the reflectance for the volume fraction. From this figure, it is found that the reflectance around 0.35 of the volume fraction changes smoothly although the relative effective permittivity is discontinuous as shown in Fig. 8.

IV. Conclusions

In this paper, the effective permittivity of the one-dimensional periodic structure has been examined numerically

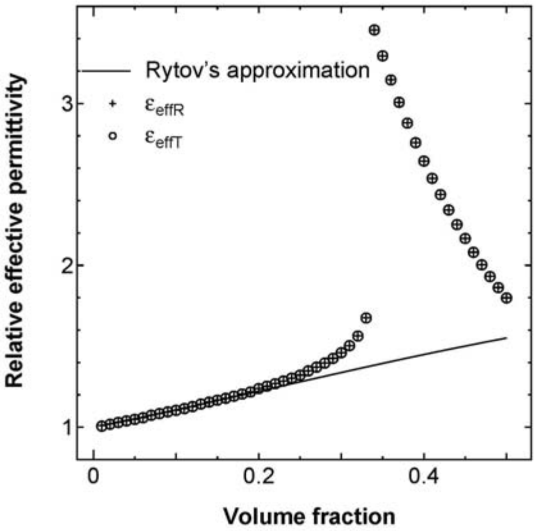

Fig. 8. Relative effective permittivity for the periodic structure of equilateral triangular cylinders.

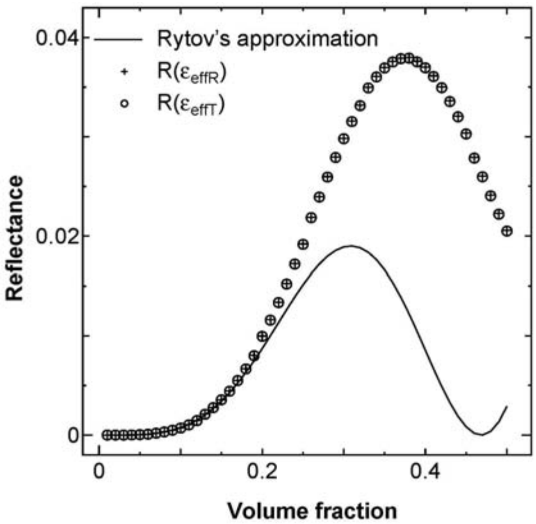

Fig. 9. Reflectance for the volume fraction from the periodic structure of equilateral triangular cylinders.

by the FDTD. The effective permittivity for the volume fraction and the normalized frequency was calculated and was compared with that obtained by Rytov approximation in order to check the validity of the presented procedure. The rectangular cylinder was well approximated by the Rytov's approximation, although other cylinders could not be approximated since the derivation of the Rytov's approximation is based on the rectangular cylinder. In addition to this, it is seen that the relative effective permittivity of the structures considered here is discontinuous at the certain point of the volume fraction. However, the reflectance by using this relative effective permittivity is continuous. The elucidation of the physical reason for this property is the future work. The effective permittivity for the multilayered structure and the effect of the angle of incidence are also the future work.

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