

Extreme parameters in systropic spherical scatterers

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Abstract—The systropic sphere is a spherical scatterer whose axes of anisotropy are defined by spherical coordinates. This contribution discusses the electrostatic scattering from a systropic sphere with extreme material parameters. The permittivity divides into three component in the spherical coordinates. Larger component values produce more polarizability, but there is a limit to the increase of the polarizability. There is also a limit to the polarizability decrease when the components become small. The goal of this contribution is to illustrate the electrostatic response of the different extreme parameter systropic spheres.

I. INTRODUCTION

A spherical shape attracts plenty of attention in the electromagnetics literature. The topic is focal for three reasons. First, a variety of natural objects take approximately spherical shape. Second, spherical objects have numerous engineering applications. Third, a spherical shape simplifies the mathematical analysis.

A particularly interesting class of spherical objects consists of onion-like structures that combine multiple tightly stratified concentric spherical layers. From the point of view of electromagnetics, such onion-like structures have anisotropic material parameters. Even though the object itself is spherically symmetric, the material responds differently to the radial electric field component than to the tangential one. Therefore the material is locally anisotropic. The dyadic representation of the relative permittivity of such a sphere is

$$\bar{\epsilon} = \epsilon_{rr}\mathbf{u}_r\mathbf{u}_r + \epsilon_{\theta\theta}\mathbf{u}_\theta\mathbf{u}_\theta + \epsilon_{\varphi\varphi}\mathbf{u}_\varphi\mathbf{u}_\varphi \quad (1)$$

where the two tangential permittivity components are equal $\epsilon_t = \epsilon_{\theta\theta} = \epsilon_{\varphi\varphi}$. The sphere with this material parameter profile has been labeled radially uniaxial (RU) [1]. The ratio between the radial and the tangential permittivity components is called the anisotropy ratio $AR = \epsilon_{rr}/\epsilon_t$. Thus, an isotropic sphere is an RU-sphere with $AR = 1$. The radar cross section of an RU-sphere can be calculated analytically [2]. The RU-sphere has also been introduced in electrostatics [3], [4].

The concept of the RU-sphere can be generalized to include spherical objects that have two separate tangential permittivity components. The members of this broader class of spherical scatterers have been labeled *systropic*. The word “systropic” originates from the Greek word “συστροφοφής” meaning “to twist”. In the present use, the word “systropic” refers to a material whose axis of anisotropy is defined by the spherical coordinates. The ratio between the two tangential components is called the systropy ratio $SR = \epsilon_{\varphi\varphi}/\epsilon_{\theta\theta}$.

The electrical response of a systropic sphere is more complex than that of an RU-sphere. The added complexity

originates from the different symmetry properties of these two scatterers. The RU-sphere is spherically symmetric whereas the systropic sphere is rotationally symmetric with respect to the z -axis. Assuming a static and uniform excitation field $\mathbf{E}^P(\mathbf{r})$, the electrical response of the systropic sphere can be expressed in terms of a dipole field $\mathbf{E}_d(\mathbf{r})$ in the far-region of the sphere. The factor of proportionality between the electric excitation field \mathbf{E}^P and the dipole moment \mathbf{p} corresponding to the dipole field $\mathbf{E}_d(\mathbf{r})$ is the polarizability of the scatterer

$$\mathbf{p} = \bar{\alpha} \cdot \mathbf{E}^P \quad (2)$$

Denoting the volume of the sphere by V and the absolute permittivity by ϵ_0 , the following normalization is adopted to obtain a dimensionless quantity

$$\bar{\alpha}_n = \frac{\bar{\alpha}}{\epsilon_0 V} \quad (3)$$

The proportionality is expressed in terms of a dyadic because the magnitude of the electric response may depend on the orientation of the excitation field \mathbf{E}^P due to the possible lack of spherical symmetry. On the other hand, the polarizability of the RU-sphere is a scalar.

The polarizability of the systropic sphere divides into two components: one for the excitation field \mathbf{E}_\parallel that is parallel to the axis of symmetry of the scatterer and one for the transverse excitation field \mathbf{E}_\perp . Thus, the polarizability dyadic can be written as

$$\bar{\alpha}_n = \alpha_{n,\parallel}\mathbf{u}_z\mathbf{u}_z + \alpha_{n,\perp}(\bar{\mathbf{I}} - \mathbf{u}_z\mathbf{u}_z) \quad (4)$$

The rotational symmetry implies that the systropic sphere virtually reduces to an RU sphere when the parallel excitation field is applied. The permittivity component $\epsilon_{\varphi\varphi}$ becomes irrelevant, because the total electric field inside the scatterer is perpendicular to \mathbf{u}_φ . In consequence, the parallel polarizability can be solved analytically.

The transverse polarizability $\alpha_{n,\perp}$ proves more intricate. The orientation of the excitation field breaks the rotational symmetry of the scattering problem allowing the component $\epsilon_{\varphi\varphi}$ to contribute to the perturbation field. An analytical solution for the transverse polarizability $\alpha_{n,\perp}$ is difficult to obtain. Instead, the problem can be solved semi-analytically.

II. GENERALIZED LAPLACE EQUATION

The Laplace equation for sourceless anisotropic medium is

$$\nabla \cdot (\bar{\epsilon} \cdot \nabla \phi) = 0 \quad (5)$$

Substituting the material parameters (1) and expressing the ∇ -operator in spherical coordinates gives

$$\frac{\epsilon_{rr}}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{\epsilon_{\theta\theta}}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left(\sin^2 \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{\epsilon_{\varphi\varphi}}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2} = 0 \quad (6)$$

This differential equation can be separated into three parts. The most intricate one is the equation for the θ -dependence. Writing $x = \cos \theta$ the equation becomes

$$(1 - x^2)g''(x) - 2xg'(x) + \left[(\mu + l)(\mu + l + 1) - \frac{\mu^2}{1 - x^2} \right] g(x) = 0, \quad -1 \leq x \leq 1 \quad (7)$$

where $\mu = \sqrt{\epsilon_{\varphi\varphi}/\epsilon_{\theta\theta}}$ and $l = 0, 1, 2, \dots$ is a separation constant. A finite and symmetric solution for this equation is

$$g(x) = (1 - x^2)^{\mu/2} C_{2l}^{(\mu+\frac{1}{2})}(x) \quad (8)$$

where $C_{2l}^{(\mu+\frac{1}{2})}$ is a Gegenbauer (ultraspherical) polynomial [5]. The symmetry $g(x) = g(-x)$ rules out the odd-degree Gegenbauer polynomials.

The different values for the separation constant l give different solutions $\phi_l(\mathbf{r})$ to (6). The general solution can be expressed as an infinite sum of these solutions. Because the homogeneous and isotropic material in which the sphere is embedded is a special case of the systropic material, the decomposition into infinite sums applies both inside the sphere $\phi_{\text{in}} = \sum A_l \phi_{\text{in},l}$ and outside the sphere $\phi_{\text{out}} = \sum B_l \phi_{\text{in},l}$. The coefficients A_l and B_l are determined by the interface conditions $\phi_{\text{in}} = \phi_{\text{out}}$ and $\epsilon_{rr} \partial_r \phi_{\text{in}} = \partial_r \phi_{\text{out}}$ applied on the surface of the sphere. Because the infinite sums must be cut into finite ones and the resulting equation group gives only an approximative solution for the field ϕ_{out} , the method does not provide an analytical solution except for the readily available solution of the RU-sphere. The semi-analytical approximation, however, is reliable for a wide range of material parameters.

III. RESULTS

The described method allows the computation of the transverse polarizability $\alpha_{n,\perp}$ for a wide range of material parameter values. Figure 1 shows the results for constant $\epsilon_{\varphi\varphi}$ values and varying ϵ_{rr} and $\epsilon_{\theta\theta}$. It stands to reason that high permittivity values cause a high polarizability and low permittivity values cause a low polarizability. It is also rather intuitive that the polarizability has the upper and lower bounds. The upper bound turns out to be $\alpha_{n,\perp} = 3$, corresponding to the PEC-sphere, and the lower bound $\alpha_{n,\perp} = -3/2$, corresponding to the PMC-sphere. However, the upper bound is not reached if $\epsilon_{\varphi\varphi}$ vanishes. Instead, the polarizability remains negative regardless of the values ϵ_{rr} and $\epsilon_{\varphi\varphi}$. If the component $\epsilon_{\varphi\varphi}$ would deviate from zero even slightly, large ϵ_{rr} and $\epsilon_{\varphi\varphi}$ could enforce a PEC-sphere.

It seems that large values of the components ϵ_{rr} , $\epsilon_{\theta\theta}$, and $\epsilon_{\varphi\varphi}$ drive the polarizability towards that of a PEC-sphere,

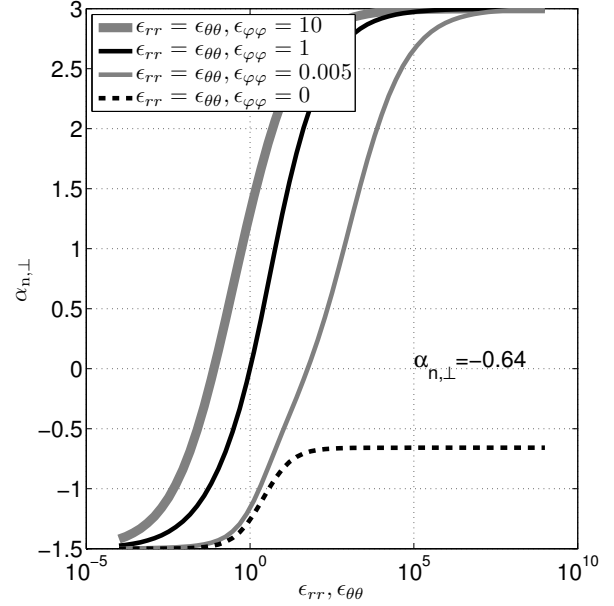


Figure 1. Polarizability curve of a scatterer with small or vanishing $\epsilon_{\varphi\varphi}$. The lowest curve deviates markedly from the rest because it displays only negative polarizability.

whereas small component values drive the polarizability towards that of a PMC-sphere. However, the paradoxical result of Figure 1 shows that the polarizability can have some strange properties in these extremes. It is thus important to consider the behavior of the polarizability $\alpha_{n,\perp}$ when extreme components are involved. The extreme values are easier to analyze for the RU-sphere than for the general systropic sphere. Analytical solution for the scattering of the general systropic sphere is still missing, but the extreme values of an RU-sphere can be found by manipulating an analytic formula.

The polarizability of the RU-sphere is given by

$$\alpha_{n,\text{RU}} = 3 \frac{\epsilon_r + 2 - \epsilon_r \sqrt{1 + 8 \frac{\epsilon_t}{\epsilon_r}}}{\epsilon_r - 4 - \epsilon_r \sqrt{1 + 8 \frac{\epsilon_t}{\epsilon_r}}} \quad (9)$$

where $\epsilon_r = \epsilon_{rr}$ and $\epsilon_t = \epsilon_{\theta\theta} = \epsilon_{\varphi\varphi}$ are the material parameters of the RU-sphere. The following values for polarizabilities α_n can be obtained when either one or both of the components ϵ_r and ϵ_t have special values:

$$\left\{ \begin{array}{ll} \alpha_n = 3 \frac{\epsilon - 1}{\epsilon + 2}, & \epsilon_r = \epsilon_t = \epsilon \\ \alpha_n = -\frac{3}{2}, & \epsilon_r \rightarrow 0, \quad 0 < \epsilon_t < \infty \\ \alpha_n = \frac{3(2\epsilon_t - 1)}{2(\epsilon_t + 1)}, & \epsilon_r \rightarrow \infty, \quad 0 < \epsilon_t < \infty \\ \alpha_n = -\frac{3}{2}, & \epsilon_t \rightarrow 0, \quad 0 < \epsilon_r < \infty \\ \alpha_n = 3, & \epsilon_t \rightarrow \infty, \quad 0 < \epsilon_r < \infty \end{array} \right. \quad (10)$$

There is a notable asymmetry between the two components. A

PEC-like sphere can be enforced by choosing $\epsilon_t \rightarrow \infty$ but not by choosing $\epsilon_r \rightarrow \infty$. To produce a PMC-like sphere, either one of the two components ϵ_r, ϵ_t has to vanish.

It is worth noting in (10) that the limit of the polarizability α_n would become ill-defined if the two components would approach opposite extremes, the ϵ_r component becoming increasingly large and the ϵ_t component becoming diminishingly small. In that case

$$\lim_{\epsilon_r \rightarrow 0} \left(\lim_{\epsilon_t \rightarrow \infty} \alpha_n \right) = 3 \quad (11)$$

$$\lim_{\epsilon_t \rightarrow \infty} \left(\lim_{\epsilon_r \rightarrow 0} \alpha_n \right) = -\frac{3}{2} \quad (12)$$

The above equations show that the limit is not commutative. The order in which the two limits are applied decides between a PEC-like and a PMC-like polarizability. The other combination of extremes, $\epsilon_r \rightarrow \infty$ and $\epsilon_t \rightarrow 0$, is less troublesome. The limits can be taken in an arbitrary order:

$$\lim_{\epsilon_r \rightarrow \infty} \left(\lim_{\epsilon_t \rightarrow 0} \alpha_n \right) = \lim_{\epsilon_t \rightarrow 0} \left(\lim_{\epsilon_r \rightarrow \infty} \alpha_n \right) = -\frac{3}{2} \quad (13)$$

One way to alleviate the problem of ill-defined limits is to replace the double limit with a single limit that changes the two parameters simultaneously. Using the geometric average of the two components $\epsilon = (\epsilon_r \epsilon_t)^{1/2}$ and the anisotropy ratio $AR = \epsilon_r / \epsilon_t$, the polarizability can be expressed as

$$\alpha_n = 3 \frac{\epsilon \sqrt{AR} + 2 - \epsilon \sqrt{AR + 8}}{\epsilon \sqrt{AR} - 4 - \epsilon \sqrt{AR + 8}} \quad (14)$$

If the anisotropy ratio is finite, the polarizability α_n has the following limits

$$\lim_{\epsilon \rightarrow \infty} \alpha_n = 3 \quad (15)$$

$$\lim_{\epsilon \rightarrow 0} \alpha_n = -\frac{3}{2} \quad (16)$$

If, on the other hand, the geometric average ϵ is fixed, the following limits can be found

$$\lim_{AR \rightarrow \infty} \alpha_n = -\frac{3}{2} \quad (17)$$

$$\lim_{AR \rightarrow 0} \alpha_n = 3 \frac{\sqrt{2}\epsilon - 1}{\sqrt{2}\epsilon + 2} \quad (18)$$

The electrical response can also be calculated for the general systropic sphere in the extreme parameter cases. As before, it is necessary to change all the parameters in unison in order to obtain a well defined limit. Defining a parameter M , which represents a large positive real number, each of the parameters ϵ_{rr} , $\epsilon_{\theta\theta}$, and $\epsilon_{\varphi\varphi}$ gets either the value M or M^{-1} . The results of these polarizability calculations are summarized in Table I. The question mark signifies the polarizability values that remain unknown in the light of present research.

Table I
COMPETING EXTREME PERMITTIVITIES WHEN $M \rightarrow \infty$

ϵ_{rr}	$\epsilon_{\theta\theta}$	$\epsilon_{\varphi\varphi}$	$\alpha_{n,\perp}$
M^{-1}	M^{-1}	M^{-1}	$-3/2$
M^{-1}	M^{-1}	M	?
M^{-1}	M	M^{-1}	-1.23
M^{-1}	M	M	0.364
M	M^{-1}	M^{-1}	$-3/2$
M	M^{-1}	M	?
M	M	M^{-1}	0.456
M	M	M	3

IV. CONCLUSION

This contribution analyzed the scattering properties of extreme parameter systropic spheres. The results for RU-spheres were considered first. It turned out that extreme parameter RU-spheres could be analyzed by using an analytical formula. The results for the general systropic sphere proved more difficult to produce. There were combinations of extreme material parameters that could not be analyzed with sufficient confidence. The systropic sphere was analyzed only in some special cases.

The systropic sphere is a spherical scatterer, which is not spherically symmetric due to the complex material parameters. As such it provides insights about the effects of stark material parameter contrasts. It is important to analyze this sort of contrasts analytically because the numerical survey of extreme parameter scatterers is notoriously difficult. Semi-analytical results for the systropic spheres can show how accurate or inaccurate the numerical descriptions are. They also offer a glimpse of the paradoxical character of the extreme parameter materials.

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