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The Far Field Asymptotics in Diffraction by a Plane Sector

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***Abstract*—In this work we study the problem of diffraction of an acoustic plane wave by a planar angular sector with the Dirichlet boundary condition on its surface. By means of the incomplete separation of variables, with the aid of the Watson-Bessel integral representation the problem is reduced to an infinite system of linear summation equations of the second kind. Exploiting the reduction of the integral representation to that of the Sommerfeld type, a consequent procedure is then developed in order to describe different components in the far field asymptotics. To that end, the analytic properties and singularities of the integrand in the Sommerfeld integral are carefully studied. The latter play a crucial role when evaluating the Sommerfeld integral by means of the saddle point technique, because these singularities are captured in the process of deformation of the Sommerfeld contours into the steepest descent paths. The corresponding asymptotic contributions of the singularities lead to description of the different types of waves in the far field asymptotics. These are the waves reflected from the sector, the waves from the edges including those multiply diffracted from one edge to another. The spherical wave from the vertex of the sector is specified by the saddle points. The singularities migrate, provided the observation point moves, and may coalesce with each other and with the saddle points, which requires more accurate asymptotic evaluation of the Sommerfeld integral in terms of the transition special functions closely related to the Fresnel integral.**

I. Introduction

The work is devoted to a new approach developed with the aim of description of the far field asymptotics in the problem of diffraction by a plane angular sector. It is obvious that a sector is an example of a degenerate elliptic cone. Consider unit sphere with the center at the sector's vertex then the sector and the sphere are intersected across a segment AB of a big circle, Fig. 1. We assume that the angular measure $2a$ of the corresponding arc satisfies the restrictions $0 < 2a < \pi$. One of the most interesting cases is the quarter-plane corresponding to $2a = \pi/2$.

It seems that study of diffraction by a plane sector began by considering this problem as a limiting case of diffraction by an elliptic cone [1] when one axis shrinks into a point. The exact and numerical solution is studied in the work [2] (see also [3]). The solution given in the work [4] is discussed in [5], where it is claimed to be erroneous. Some recent results are connected with calculation of the diffraction coefficient of the spherical wave for a quarter-plane [6],[7] that was developed in [8]. The formulas for the scattered waves are obtained in the paper [9], where the authors apply an approach based on a set physical postulates which are closely connected with the well-known localization principle in short-wave length diffraction theory. Analogous results are discussed in the work [10] except doubly diffracted edge waves. The approach utilized in the paper [11] is, in a sense, close to that used in the present work. After separation of the radial variable the author of [11] studies the leading terms of the asymptotics with respect to the large spectral parameter for the 'stationary' problem on the unit sphere with the cut. In our work we also use a similar ideology, however, we prefer to apply a 'non-stationary' version of the diffraction theory in a problem for hyperbolic equation on the unit sphere. In the framework of the Sommerfeld integral formalism we carefully investigate real singularities of the unknown function (Sommerfeld transformant) in the intergand. These singularities are responsible for different components in the far field obtained from asymptotic evaluation of the Sommerfeld integral. Actually the 'stationary' approach of [11] and that exploited in this work are connected by the Fourier transform.

II. Description of the Approach

In the following we formulate the problem of diffraction of an acoustic plane wave by a plane sector with the angular opening $2a$. The wave field satisfies the Helmholtz equation, the Dirichlet boundary conditions, the Meixner's conditions at the edges and that at the vertex.

In order to formulate conditions at infinity we, in the next section, introduce a set of characteristic domains of the unit sphere centered at the vertex of the sector. Each point of such a domain on the sphere is attributed to some direction of observation. In different characteristic domains the far field asymptotics consists of different wave field components. Near the common boundary of two adjacent domains the asymptotics is described by a corresponding transition special function which appropriately matches local asymptotic expressions in these domains. Sometimes such directions are called singular ones because diffraction coefficients (or, in other words, scattering amplitudes) in the local asymptotics are singular there.

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One of the main goals of this work is description of a method (see Chapters 5,6 in [12], where it is applied for an impedance cone) which enables one to give the far field asymptotic expressions for the problem.

To that end, we use the Watson-Bessel integral representation for the solution in order to separate radial variable and to formulate the problem for the unknown 'spectral' function on the unit sphere with the cut AB. By means of further separation of the spherical variables we arrive at the summation linear system of the second kind for the corresponding Fourier coefficients of the series for the spectral function. In order to compute the far field asymptotics the Watson-Bessel representation of the solution is then reduced to the Sommerfeld integral, whereas the Sommerfeld transformant is nothing but the Fourier transform of the spectral function.

The Sommerfeld integral is asymptotically computed in order to obtain the far field expressions. To that end, the Sommerfeld contours are deformed into the steepest descent paths (SDP) passing through the saddle points $\pm\pi$. In the process of such deformation the singularities of the transformants can be captured. These singularities are poles or branch points. Location of such singularities depends upon the observation point and they migrate with the variation of the observation direction. Contribution of the saddle points gives rise to the spherical wave from the vertex, whereas the contributions from the singularities are responsible for the reflected or other diffracted waves. The directions in which the fronts of two or more waves are tangent correspond to the singular directions. In these directions the singularities coalesce with each other or (and) with the saddle points. The asymptotics is described by special transition functions in this case. As a result, an essential part of the analysis deals with the study of the singularities.

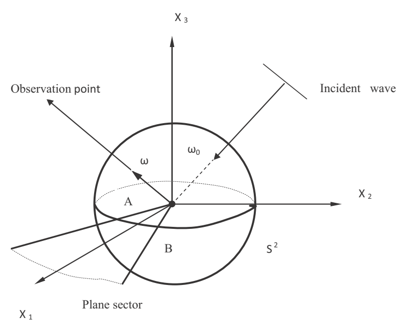

Fig. 1. Diffraction by a sector

A. Formulation of the Problem

Introduce the spherical coordinates (r, ϑ, φ) attributed to the Cartesian ones by the correlations

$$X_1 = r \cos\varphi \sin\vartheta,\ X_2 = r \sin\varphi \sin\vartheta,\ X_3 = r \cos\vartheta.$$

A plane wave is incident from the direction specified by $\omega_0 = (\vartheta_0, \varphi_0)$ (Fig. 1.1)

$$U_i(r, \vartheta, \varphi) = \exp\{-ikr \cos\theta_i(\omega, \omega_0)\}, \qquad (1)$$

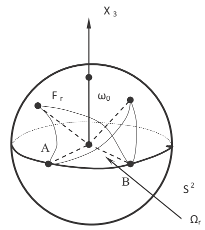

Fig. 2. The triangular domain Ω_r with the vertexes A, B, F_r on S^2

where $\omega = (\vartheta, \varphi)$ corresponds to the direction of observation and $\cos\theta_i(\omega, \omega_0) = \cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos[\varphi - \varphi_0]$.[1] The total wave field $U(r, \vartheta, \varphi) + U_i(r, \vartheta, \varphi)$ is the sum of the scattered and incident fields,

$$(\triangle + k^2)U(r, \vartheta, \varphi) = 0, \qquad (2)$$

$k > 0$ is the wave number. The Dirichlet boundary condition

$$(U_i + U)|_S = 0, \qquad (3)$$

is satisfied on the sector S, $S = \{(r, \omega) : r \geq 0, \omega \in AB\}$. Sometimes we shall denote the closed arc AB by σ, $\sigma = S \cap S^2$. The Meixner's edge condition is postulated near the edges $\partial S_{1,2}$ (and outside the close vicinity of the vertex)

$$U \sim C_{1,2}(z)\rho^{1/2},\ \rho \to 0 \qquad (4)$$

uniformly with respect to ϕ, where ρ, ϕ, z are natural local cylindrical coordinates attributed to the edges $\partial S_{1,2}$ which we also denote A, B. The conditions at the vertex of the sector read

$$|U| \leq C\,r^{-1/2},\ |\nabla U| \leq C\,r^{-3/2},\ r \to 0 \qquad (5)$$

which are valid uniformly with respect to the angular variables. Description of far field behavior is the main goal of the paper.

B. The far field behavior

In order to give a detailed description of the far field components in the scattered wave it is reasonable to use some simple geometrical constructions and to define some subdomains on the unit sphere S^2 with the cut $\sigma = \overline{AB}$. Consider the geodesic distance $\theta(\omega, \omega_0)$ between two points ω and ω_0 on the sphere S^2 which satisfies the eikonal equation

$$(\nabla_\omega \theta(\omega, \omega_0))^2 = 1, \qquad (6)$$

where $\nabla_\omega = \vec{e}_\vartheta \frac{\partial}{\partial\vartheta} + \vec{e}_\varphi \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}$ is the gradient operator on the unit sphere. It is obvious that $\theta_i(\omega, \omega_0)$ defined above coincides with $\theta(\omega, \omega_0)$.

[1]The harmonic time-dependence $e^{-i\widehat{\omega}t}$ is assumed and suppressed throughout the paper.

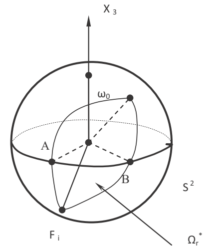

Fig. 3. The triangular domain Ω_r^* with the vertexes ABF_i on S^2

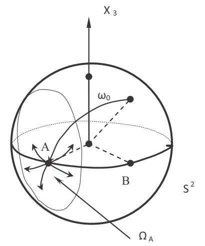

Fig. 4. The circular domain Ω_A on S^2

In the same manner, introduce the 'broken' geodesic $\theta_r(\omega, \omega_0)$,

$$\theta_r(\omega, \omega_0) = \min_{l \in \sigma}(\theta(\omega, l) + \theta(l, \omega_0)) \tag{7}$$

which fulfills the equation

$$(\nabla_\omega \theta_r(\omega, \omega_0))^2 = 1. \tag{8}$$

The geodesic corresponding to the solution (7) of the equation (8) has simple geometrical meaning: this is a broken geodesic of the minimal length which originates at the source ω_0 then reflects on the boundary $\sigma = \overline{AB}$ in accordance with geometrical optics laws and arrives at the point ω. The 'incident' parts of such broken geodesics fill in the spherical triangle $A\omega_0 B$ in Fig. 1.2, whereas the 'reflected' parts fill in the spherical triangle $F_r BA$ which is further denoted Ω_r.

In the same manner we specify the spherical triangular domain Ω_r^* coinciding with the triangle ABF_i in Fig. 1.3. This domain is bounded by σ and by the corresponding parts of two geodesics emanated at ω_0 passed through the points A and B and arrived at the common point of their intersection F_i. Remark that the length of each geodesic $\omega_0 AF_i$ and $\omega_0 BF_i$ is equal to π. It is worth mentioning that points F_i and F_r are correspondingly the points of focusing of the incident and reflected geodesics having common points with the cut σ. It is obvious that the domain Ω_r^* is the mirror image of Ω_r with respect to the boundary σ for the same fixed position of ω_0.

Now we specify additionally two other domains Ω_A (see Fig. 1.4) and Ω_B. Consider the ray (geodesic) $\omega_0 A$ which arrives at the edge point A and produces a set of 'diffracted' rays (geodesics) outgoing from A in all direction. For each point ω there exists such a diffracted ray with length $\psi_A, (\psi_A < \pi)$ that arrives at this point. We define Ω_A as a circle on the sphere such that $\Omega_A = \{\omega \in S^2 : 0 \le \theta_A(\omega, \omega_0) := \theta(A, \omega_0) + \psi_A < \pi\}$. The eikonal $\theta_A(\omega, \omega_0)$ satisfies the equation

$$(\nabla_\omega \theta_A(\omega, \omega_0))^2 = 1. \tag{9}$$

In quite similar manner the domain Ω_B is specified, $\Omega_B = \{\omega \in S^2 : 0 \le \theta_B(\omega, \omega_0) := \theta(\omega_0, B) + \psi_B < \pi\}$, i.e. it is a circle on S^2 centered at B with $\theta_B(\omega, \omega_0)$ satisfying the equation

$$(\nabla_\omega \theta_B(\omega, \omega_0))^2 = 1. \tag{10}$$

ψ_B is the length of the geodesic emanated from the point B.

The spherical domains Ω_A and Ω_B intersect with Ω_r and Ω_r^* and with each other. The domain Ω_r^* corresponds to directions in which geometrical shadow of the incident wave is observed. Ω_r forms a set of directions of propagation of the space rays reflected from the sector S. The directions specified by Ω_A (or by Ω_B) correspond to the directions in which the diffracted from the edge A (or from B) space cylindrical wave is observed. These simple facts follow from the analysis represented below in this paper.

Remark. It is worth commenting that we can also introduce domain Ω_{AB} (and analogously Ω_{BA}) which corresponds to all 'rays' that originate from ω_0, arrive at the point A going along the arc AB to the edge point B then along a geodesic arrive at ω. The length $\theta_{AB}(\omega, \omega_0) := \theta(\omega_0, A) + 2a + \psi_B$ of such a compound geodesic should be less than π, which specifies Ω_{AB}. The corresponding space wave is the cylindrical wave from the edge A propagating to the edge B and diffracted there and arrive at the observation point. The domains $\Omega_{ABA}, \Omega_{BAB}, \Omega_{ABAB} \dots$, etc. are defined and interpreted quite analogously.

III. Basic Results

We consider the domain $\Omega_0 = S^2 \setminus (\Omega_A \cup \Omega_B \cup \Omega_r)$ on the sphere S^2, which is called 'oasis'.[2] The scattered far field (total minus incident) in this domain of directions consists of the spherical wave propagating from the vertex of the sector

$$U(r, \vartheta, \varphi) = D(\omega, \omega_0) \frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr} \left(1 + O\left(\frac{1}{kr}\right)\right), \quad kr \to \infty. \tag{11}$$

In the non-uniform with respect to the observation angles asymptotics (11) the diffraction coefficient $D(\omega, \omega_0)$ is to be determined from the further analysis and is one of the most important characteristics of the scattered field. The asymptotics (11) fails provided the observation point approaches the boundary of Ω_0. As it has been already remarked near the boundaries of the domains $\Omega_s,\ s = 0, r, A, B, \dots$ some special transition functions apply to match the local asymptotics.

In the exterior of the oasis the structure of asymptotics is more complex and contains also other wave components in the far field. Consider the directions from $(S^2 \setminus (\Omega_r^* \cup \Omega_r \cup \Omega_B)) \cap \Omega_A$ in which the spherical wave and the cylindrical

[2]Remark that Ω_r^* and Ω_r are covered by $\Omega_A \cup \Omega_B$.

wave from the edge A (as well as possibly multiply diffracted) are observed in the far field

$$U(r,\vartheta,\varphi) = D(\omega,\omega_0)\frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\left(1+O\left(\frac{1}{kr}\right)\right)+$$
$$d_A(\omega,\omega_0)\,\frac{\exp(-\mathrm{i}kr\cos\theta_A(\omega,\omega_0))}{\sqrt{-\mathrm{i}kr\sin\psi_A}}\left(1+O\left(\frac{1}{kr\sin\psi_A}\right)\right)$$
$$+\dots, \tag{12}$$

where dots denote the multiply diffracted from the edges waves, provided the corresponding directions belong also to $\Omega_{BA},\Omega_{ABA},\dots$.[3] The unknown yet function $d_A(\omega,\omega_0)$ in (12) is connected with the diffraction coefficient of the cylindrical wave from the edge A.

In the domain $(S^2\setminus(\Omega_r^*\cup\Omega_r\cup\Omega_A))\cap\Omega_B$ the asymptotics has the same form as in (12) with the change of the subscript A on to B in the second summand which the describes the cylindrical wave from the edge B.

In the directions ω from $\Omega_r\cap(\Omega_B\cup\Omega_A)$ the leading terms consist of the reflected, spherical and diffracted from the edges A and B waves

$$U(r,\vartheta,\varphi) = R\exp(-\mathrm{i}kr\cos\theta_r(\omega,\omega_0))+$$
$$D(\omega,\omega_0)\frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\left(1+O\left(\frac{1}{kr}\right)\right)+$$
$$d_A(\omega,\omega_0)\,\frac{\exp(-\mathrm{i}kr\cos\theta_A(\omega,\omega_0))}{\sqrt{-\mathrm{i}kr\sin\psi_A}}\left(1+O\left(\frac{1}{kr\sin\psi_A}\right)\right)+$$
$$d_B(\omega,\omega_0)\,\frac{\exp(-\mathrm{i}kr\cos\theta_B(\omega,\omega_0))}{\sqrt{-\mathrm{i}kr\sin\psi_B}}\left(1+O\left(\frac{1}{kr\sin\psi_B}\right)\right)+$$
$$\dots, \tag{13}$$

where $R = -1$ in the first summand $U_r(\omega,\omega_0) = R\exp(-\mathrm{i}kr\cos\theta_r(\omega,\omega_0))$ of (13) which is the reflected wave.

The total wave field $U+U_i$ in the shadow of the incident wave, i.e. as $\omega\in\Omega_r^*\cap(\Omega_B\cup\Omega_A)$, reads

$$U(r,\vartheta,\varphi)+U_i(r,\vartheta,\varphi) =$$
$$D(\omega,\omega_0)\frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\left(1+O\left(\frac{1}{kr}\right)\right)+$$
$$d_A(\omega,\omega_0)\,\frac{\exp(-\mathrm{i}kr\cos\theta_A(\omega,\omega_0))}{\sqrt{-\mathrm{i}kr\sin\psi_A}}\left(1+O\left(\frac{1}{kr\sin\psi_A}\right)\right)+$$
$$d_B(\omega,\omega_0)\,\frac{\exp(-\mathrm{i}kr\cos\theta_B(\omega,\omega_0))}{\sqrt{-\mathrm{i}kr\sin\psi_B}}\left(1+O\left(\frac{1}{kr\sin\psi_B}\right)\right)$$
$$+\dots, \tag{14}$$

$kr\to\infty$.

It is worth commenting that in the direction corresponding to the points F_r ($\theta_r\sim\pi,\theta_A\sim\pi,\theta_B\sim\pi$) or F_i ($\theta_i\sim\pi,\theta_A\sim\pi,\theta_B\sim\pi$) the wave field behavior is the most complex because in this direction at least two transition regions intersect. In this directions the wave fronts of the spherical, reflected (or incident) and edge waves are tangent. As we show below this corresponds to approaching the corresponding singularities of the Sommerfeld transformants to the saddle point $\pm\pi$. Some results on the far field behavior in these directions can be found in [11].

In the report we intend to give a systematic procedure in order to demonstrate the anticipated asymptotics in (11)–(14) and to obtain formulas for the diffraction coefficients.

[3] Usually these waves are neglected in comparison with the first two terms.

IV. Conclusion

In this work we developed a self-consistent procedure in order to derive the far field asymptotics in the problem of diffraction by a plane sector. It is based on flexible use of the integral transforms of the solution, study of its analytic properties, namely, of the singularities of the Sommerefeld transformants. Further use of the saddle point technique enabled us to derive the desired asymptotics. The similar ideas have been exploited for the description of the far field scattered by an impedance cone [12]. It is remarkable that the approach can be additionally modified and adapted to the problem of diffraction by a pyramidal cone so that the problem at hand is an important step towards this goal.

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