Scattering by inhomogeneities in parallel plate waveguides

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***Abstract*— Electromagnetic scattering by a bounded object located inside a parallel plate waveguide is the subject of this paper. The exciting field in the waveguide is either an arbitrary source located at a finite distance from the obstacle or a plane wave generated in the far zone. The analytic treatment of the problem relies on an extension of the null field approach or T-matrix method.**

I. Introduction

Recent theoretical progress in the development of useful scattering identities — sum rules [1] – have initiated several attempts to verify these identities experimentally, see *e.g.*, [2]. These sum rules relate the dynamical behavior of the scattering and absorption behavior of the scatterer to the static properties of the scatterer (polarizability dyadics).

Initial investigations show that the parallel plate waveguide is accessible [2, 3]. A detailed investigation of the static properties of an obstacle between two parallel plates has also been reported recently [4].

The approach employs the integral representation of the solution. This integral representation approach to solve the scattering problem was originally introduced by Peter Waterman for finite scatterers, and it has proven to be a very powerful and useful technique to solve a large variety of scattering problems, not only electromagnetic, but also acoustic and elastodynamic problems. In fact, the present geometry is an extension of the results with buried obstacle close to a planar interface — layered or not, see *e.g.*, [5, 6].

II. Formulation of the problem

A finite scatterer with bounding surface S_s defines the region V_s. Two infinite, perfectly conducting planes, S_+ and S_-, confine the regions V_e and V_s, see Figure 1. These planes are parameterized by $z = z_+$ and $z = z_-$, respectively. The regions above S_+ and below S_- are denoted by V_+ and V_-, respectively. The sources of the problem are assumed to be locates in $V_\mathrm{i} \subset V_\mathrm{e}$, between the surfaces S_+ and S_-.

To proceed, the time-harmonic electric and magnetic fields satisfy the free-space Maxwell equations in V_e (we use the time convention $\exp\{-\mathrm{i}\omega t\}$),

$$\begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{r}) = \mathrm{i}k_0\eta_0 \boldsymbol{H}(\boldsymbol{r}) \\ \nabla \times \eta_0 \boldsymbol{H}(\boldsymbol{r}) = -\mathrm{i}k_0 \boldsymbol{E}(\boldsymbol{r}) \end{cases} \qquad \boldsymbol{r} \in V_\mathrm{e} \tag{1}$$

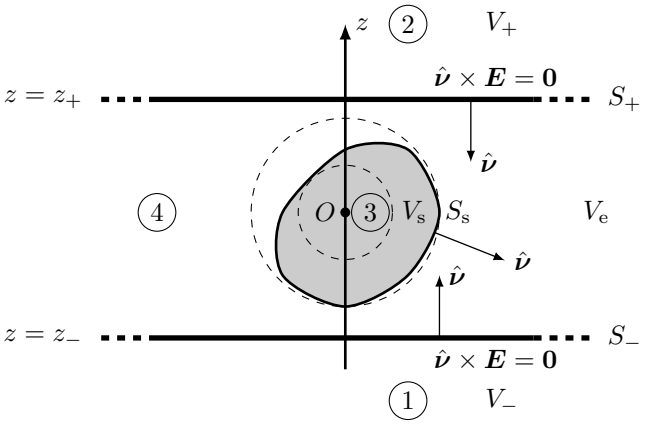

Fig. 1. The geometry of the direct scattering problem with two perfectly conducting planes S_+ and S_- and a scatterer with bounding surface S_s.

where $k_0 = \omega/c_0$ and η_0 are the wave number and wave impedance in free space, respectively. The boundary conditions on the bounding surfaces are

$$\begin{cases} \hat{\boldsymbol{z}} \times \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0}, & \boldsymbol{r} \in S_+ \cup S_- \\ \hat{\boldsymbol{\nu}} \times \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0}, & \boldsymbol{r} \in S_\mathrm{s} \end{cases} \tag{2}$$

The scatterer V_s is here assumed to be a perfectly conducting body. This assumption can easily be relaxed.

A. Integral representation of the solution

Let $\boldsymbol{E}_\mathrm{i}$ denote the incident electric field with sources located in V_e, and define the scattered electric field $\boldsymbol{E}_\mathrm{s} = \boldsymbol{E} - \boldsymbol{E}_\mathrm{i}$. The incident field $\boldsymbol{E}_\mathrm{i}$ is the field with no obstacle or plates present. With the directions of the unit normals defined as in Figure 1, the solution of (1) and (2) satisfies the surface integral representation [7]

$$-\frac{1}{\mathrm{i}k_0}\nabla \times \left(\nabla \times \iint_{S_+\cup S_-\cup S_\mathrm{s}} \mathbf{G}_\mathrm{e}(k_0, |\boldsymbol{r}-\boldsymbol{r}'|) \cdot \boldsymbol{K}(\boldsymbol{r}') \, \mathrm{d}S' \right) = \begin{cases} \boldsymbol{E}_\mathrm{s}(\boldsymbol{r}), & \boldsymbol{r} \in V_\mathrm{e} \\ -\boldsymbol{E}_\mathrm{i}(\boldsymbol{r}), & \boldsymbol{r} \in V_+ \cup V_- \cup V_\mathrm{s} \end{cases} \tag{3}$$

where $\boldsymbol{K} = \hat{\boldsymbol{\nu}} \times \eta_0 \boldsymbol{H}$, and the electric Green's dyadic

$$\mathbf{G}_\mathrm{e}(k_0, |\boldsymbol{r}-\boldsymbol{r}'|) = \left(\mathbf{I}_3 + \frac{1}{k_0^2}\nabla\nabla \right) \frac{\mathrm{e}^{\mathrm{i}k_0|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

The integral representation also contains a surface integral evaluated at large lateral distances, but proper radiation conditions at large lateral distances make this integral vanish. This surface integral representation is the starting point in the null-field approach.

III. Basis functions and expansions

A. Spherical and planar vector waves

To proceed, out-going or radiating spherical vector waves, $\boldsymbol{u}_{\tau\sigma ml}(k\boldsymbol{r}) = \boldsymbol{u}_{\tau n}(k\boldsymbol{r}) = \boldsymbol{u}_n(k\boldsymbol{r})$, and regular spherical vector waves, $\boldsymbol{v}_{\tau\sigma ml}(k\boldsymbol{r})$, are employed. We adopt the definition in [8, 9].

The most appropriate basis functions to deal with the geometry of the parallel plates are planar waves. We define the dimension-less, vector-valued plane waves, $\boldsymbol{\varphi}_j^{\pm}(\boldsymbol{k}_{\rm t};\boldsymbol{r})$, $j = 1, 2$, as:

$$\begin{cases} \boldsymbol{\varphi}_1^{\pm}(\boldsymbol{k}_{\rm t};\boldsymbol{r}) = \dfrac{\hat{\boldsymbol{z}} \times \boldsymbol{k}_{\rm t}}{4\pi {\rm i} k_{\rm t}} {\rm e}^{{\rm i}\boldsymbol{k}_{\rm t}\cdot\boldsymbol{\rho} \pm {\rm i}k_z z} \\ \boldsymbol{\varphi}_2^{\pm}(\boldsymbol{k}_{\rm t};\boldsymbol{r}) = \dfrac{\mp \boldsymbol{k}_{\rm t} k_z + k_{\rm t}^2 \hat{\boldsymbol{z}}}{4\pi k_0 k_{\rm t}} {\rm e}^{{\rm i}\boldsymbol{k}_{\rm t}\cdot\boldsymbol{\rho} \pm {\rm i}k_z z} \end{cases}$$

where the transverse (tangential) wave vector and the spatial position vector in the plane are

$$\boldsymbol{k}_{\rm t} = \hat{\boldsymbol{x}} k_x + \hat{\boldsymbol{y}} k_y, \quad k_x, k_y \in \mathbb{R}, \qquad \boldsymbol{\rho} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$$

The length of the transverse wave vector is always a real number, *viz.*,

$$k_{\rm t} = |\boldsymbol{k}_{\rm t}| = \sqrt{k_x^2 + k_y^2}$$

and k_z is defined by

$$k_z = \left(k_0^2 - k_{\rm t}^2\right)^{1/2} = \begin{cases} \sqrt{k_0^2 - k_{\rm t}^2} \text{ for } k_{\rm t} < k_0 \\ {\rm i}\sqrt{k_{\rm t}^2 - k_0^2} \text{ for } k_{\rm t} > k_0 \end{cases}$$

The plus super-index denotes an exponentially decreasing inhomogeneous or evanescent wave as $z \to \infty$, and similarly for the minus super index as $z \to -\infty$. The index $j = 1$ labels the TE-waves, and $j = 2$ labels the TM-waves.

B. Green's dyadic decompositions

The Green's dyadic is decomposed in spherical vector waves [8]

$$\begin{aligned} \mathbf{G}_{\rm e}(k_0, |\boldsymbol{r} - \boldsymbol{r}'|) &= {\rm i}k_0 \sum_n \boldsymbol{v}_n(k_0\boldsymbol{r}_<)\boldsymbol{u}_n(k_0\boldsymbol{r}_>) \\ &= {\rm i}k_0 \sum_n \boldsymbol{u}_n(k_0\boldsymbol{r}_>)\boldsymbol{v}_n(k_0\boldsymbol{r}_<) \end{aligned} \quad (4)$$

where $\boldsymbol{r}_<$ ($\boldsymbol{r}_>$) is the position vector with the smallest (largest) distance to the origin, *i.e.*, if $r < r'$ then $\boldsymbol{r}_< = \boldsymbol{r}$ and $\boldsymbol{r}_> = \boldsymbol{r}'$. The summation is over the divergence-free vector spherical vector waves, $\tau = 1, 2$. Moreover, we need the decomposition of the Green's dyadic in planar vector waves [8]

$$\begin{aligned} &\mathbf{G}_{\rm e}(k_0, |\boldsymbol{r} - \boldsymbol{r}'|) \\ &\quad = 2{\rm i}k_0 \sum_{j=1,2} \iint_{\mathbb{R}^2} \boldsymbol{\varphi}_j^{\pm}(\boldsymbol{k}_{\rm t};\boldsymbol{r}) \boldsymbol{\varphi}_j^{\mp\dagger}(\boldsymbol{k}_{\rm t};\boldsymbol{r}') \frac{k_0}{k_z} \frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2} \end{aligned} \quad (5)$$

where the upper (lower) is used if $z > z'$ ($z < z'$), and where the dagger corresponds to a change $\boldsymbol{k}_{\rm t} \to -\boldsymbol{k}_{\rm t}$.

C. Transformation between solutions

To connect the spherical vector waves and the planar vector waves, we need a transformation between the two sets of solutions. The result that is relevant in the analysis below are [8, p. 183]

$$\begin{aligned} &\boldsymbol{u}_n(k_0\boldsymbol{r}) \\ &= 2 \sum_{j=1,2} \iint_{\mathbb{R}^2} B_{nj}^{\pm}(\boldsymbol{k}_{\rm t}) \boldsymbol{\varphi}_j^{\pm}(\boldsymbol{k}_{\rm t};\boldsymbol{r}) \frac{k_0}{k_z} \frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2}, \quad z \gtrless 0 \end{aligned} \quad (6)$$

where

$$\begin{aligned} &B_{nj}^{\pm}(\boldsymbol{k}_{\rm t}) \\ &\quad = {\rm i}^{-l+\tau} C_{lm} (\pm 1)^{l+m} \left\{ \mp \delta_{\tau j} \Delta_l^m(k_z/k_0) \begin{Bmatrix} \cos m\beta \\ \sin m\beta \end{Bmatrix} \right. \\ &\qquad \left. - \delta_{\tau\bar{j}} \pi_l^m(k_z/k_0) \begin{Bmatrix} -\sin m\beta \\ \cos m\beta \end{Bmatrix} \right\} \end{aligned} \quad (7)$$

and where $\boldsymbol{k}_{\rm t} = k_{\rm t}(\hat{\boldsymbol{x}}\cos\beta + \hat{\boldsymbol{y}}\sin\beta)$, C_{lm} are normalization constants [9], and

$$\begin{aligned} \Delta_l^m(t) &= -\frac{\left(1 - t^2\right)^{1/2}}{\sqrt{l(l+1)}} P_l^{m\prime}(t) \\ \pi_l^m(t) &= \frac{m P_l^m(t)}{\sqrt{l(l+1)}\left(1 - t^2\right)^{1/2}} \end{aligned} \quad (8)$$

IV. Incident electric field

The sources of the incident field are assumed to be located between the plates $S_{\pm}$ and outside the scatterer $V_{\rm s}$. Due to the completeness of the planar vector waves and the spherical vector waves, the incident electric field is assumed to have the following expansions in the three regions, $V_{\pm}$ and $V_{\rm s}$:

$$\boldsymbol{E}_{\rm i}(\boldsymbol{r}) = \sum_{j=1,2} \iint_{\mathbb{R}^2} a_j^{\pm}(\boldsymbol{k}_{\rm t}) \boldsymbol{\varphi}_j^{\pm}(\boldsymbol{k}_{\rm t};\boldsymbol{r}) \frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2}, \quad \boldsymbol{r} \in V_{\pm} \quad (9)$$

where the upper (lower) sign holds for V_+ (V_-), and

$$\boldsymbol{E}_{\rm i}(\boldsymbol{r}) = \sum_n a_n \boldsymbol{v}_n(k_0\boldsymbol{r}), \quad \boldsymbol{r} \in V_R \quad (10)$$

where V_R is a sphere of radius R that does not include the circumscribing sphere of the source region. The coefficients $a_j^{\pm}(\boldsymbol{k}_{\rm t})$ and a_n are assumed known.

V. Utilizing the surface integral representation

The position vector, $\boldsymbol{r}$, can take four different principle positions, $\boldsymbol{r} \in V_{\pm}$, $\boldsymbol{r} \in V_{\rm s}$, and $\boldsymbol{r} \in V_{\rm e}$. We now explore these possibilities. The decompositions of the Green's dyadic in spherical and planar vector waves, see (4) and (5) are now used.

When the position vector is either in V_+ or in V_-, the lower line of (3) using (4), (5), (6), and (9) yield, since the planar vector waves are linearly independent

$$a_j^{\pm}(\boldsymbol{k}_{\rm t}) = 2k_0^2\frac{k_0}{k_z}\iint\limits_{S_+\cup S_-}\boldsymbol{\varphi}_j^{\mp\,\dagger}(\boldsymbol{k}_{\rm t};\boldsymbol{r}')\cdot\boldsymbol{K}(\boldsymbol{r}')\,{\rm d}S' + 2k_0^2\frac{k_0}{k_z}\sum_n B_{nj}^{\pm}(\boldsymbol{k}_{\rm t})\iint\limits_{S_{\rm s}}\boldsymbol{v}_n(k_0\boldsymbol{r}')\cdot\boldsymbol{K}(\boldsymbol{r}')\,{\rm d}S' \tag{11}$$

When the position vector is inside the largest sphere enclosed in $V_{\rm s}$, the lower line of (3) using (4), (6), and (9) yield

$$\begin{aligned} a_n &= k_0^2\iint\limits_{S_{\rm s}}\boldsymbol{u}_n(k_0\boldsymbol{r}')\cdot\boldsymbol{K}(\boldsymbol{r}')\,{\rm d}S' \\ &+ 2k_0^3\sum_{j=1,2}\iint\limits_{\mathbb{R}^2}B_{nj}^{+}(\boldsymbol{k}_{\rm t})\iint\limits_{S_+}\boldsymbol{\varphi}_j^{+}(\boldsymbol{k}_{\rm t};\boldsymbol{r}')\cdot\boldsymbol{K}(\boldsymbol{r}')\,{\rm d}S'\,\frac{{\rm d}k_x\,{\rm d}k_y}{k_zk_0^2} \\ &+2k_0^3\sum_{j=1,2}\iint\limits_{\mathbb{R}^2}B_{nj}^{-}(\boldsymbol{k}_{\rm t})\iint\limits_{S_-}\boldsymbol{\varphi}_j^{-}(\boldsymbol{k}_{\rm t};\boldsymbol{r}')\cdot\boldsymbol{K}(\boldsymbol{r}')\,{\rm d}S'\,\frac{{\rm d}k_x\,{\rm d}k_y}{k_zk_0^2} \end{aligned} \tag{12}$$

since the regular spherical vector waves, $\boldsymbol{v}_n(k_0\boldsymbol{r})$, are linearly independent.

When the position vector is outside the circumscribing sphere of $S_{\rm s}$ in $V_{\rm e}$, the upper line of (3) using (4), (5), (6), and (9) yield

$$\begin{aligned} \boldsymbol{E}_{\rm s}(\boldsymbol{r}) &= \sum_n f_n\boldsymbol{u}_n(k_0\boldsymbol{r}) \\ &+\sum_{j=1,2}\iint\limits_{\mathbb{R}^2}f_j^{+}(\boldsymbol{k}_{\rm t})\boldsymbol{\varphi}_j^{+}(\boldsymbol{k}_{\rm t};\boldsymbol{r})\,\frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2} \\ &+\sum_{j=1,2}\iint\limits_{\mathbb{R}^2}f_j^{-}(\boldsymbol{k}_{\rm t})\boldsymbol{\varphi}_j^{-}(\boldsymbol{k}_{\rm t};\boldsymbol{r})\,\frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2} \end{aligned} \tag{13}$$

where

$$\begin{cases} f_n = -k_0^2\displaystyle\iint\limits_{S_{\rm s}}\boldsymbol{v}_n(k_0\boldsymbol{r}')\cdot\boldsymbol{K}(\boldsymbol{r}')\,{\rm d}S' \\ f_j^{\pm}(\boldsymbol{k}_{\rm t}) = -2k_0^2\dfrac{k_0}{k_z}\displaystyle\iint\limits_{S_{\mp}}\boldsymbol{\varphi}_j^{\mp\,\dagger}(\boldsymbol{k}_{\rm t};\boldsymbol{r}')\cdot\boldsymbol{K}(\boldsymbol{r}')\,{\rm d}S' \end{cases} \tag{14}$$

VI. Expansion and Elimination of the Surface Fields

Expand the currents on the surfaces in planar vector waves and a complete set of tangential vector functions, $\hat{\boldsymbol{\nu}}\times\boldsymbol{\psi}_n$, on $S_{\rm s}$. We assume

$$\boldsymbol{K}(\boldsymbol{r}) = \sum_{j=1,2}\iint\limits_{\mathbb{R}^2}\alpha_j^{\pm}(\boldsymbol{k}_{\rm t})\hat{\boldsymbol{z}}\times\boldsymbol{\varphi}_{\overline{j}}^{\mp}(\boldsymbol{k}_{\rm t};\boldsymbol{r})\,\frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2},\quad \boldsymbol{r}\in S_{\pm}$$

and

$$\boldsymbol{K}(\boldsymbol{r}) = \sum_n\alpha_n\hat{\boldsymbol{\nu}}(\boldsymbol{r})\times\boldsymbol{\psi}_n(\boldsymbol{r}),\qquad \boldsymbol{r}\in S_{\rm s}$$

where the dual index $\overline{j}$ is defined $\overline{1}=2$ and $\overline{2}=1$.

Insert these expansions in (11) and (12), use the orthogonality of the planar vector waves and solve for the unknown coefficients $\alpha_j^{\pm}(\boldsymbol{k}_{\rm t})$ and α_n. The result is

$$\begin{aligned} \alpha_j^{\pm}(\boldsymbol{k}_{\rm t}) &= \mp 2{\rm i}\frac{a_j^{\mp}(\boldsymbol{k}_{\rm t})+a_j^{\pm}(\boldsymbol{k}_{\rm t})(-1)^j{\rm e}^{\pm 2{\rm i}k_zz_{\mp}}}{1-{\rm e}^{-2{\rm i}k_zd}} \\ &\mp 4{\rm i}\frac{k_0}{k_z}\sum_{nn'}\frac{B_{nj}^{\mp}(\boldsymbol{k}_{\rm t})+B_{nj}^{\pm}(\boldsymbol{k}_{\rm t})(-1)^j{\rm e}^{\pm 2{\rm i}k_zz_{\mp}}}{1-{\rm e}^{-2{\rm i}k_zd}} \\ &\qquad\times T_{nn'}\left(a_{n'}+\gamma_{n'}\right) \end{aligned} \tag{15}$$

where the distance between the plates is $d=z_+-z_-$, the T-matrix, $T_{nn'}$, of the scatterer is defined in [10], and

$$\gamma_n = \frac{1}{2{\rm i}}\sum_{j=1,2}\iint\limits_{\mathbb{R}^2}\left(\alpha_j^{+}(\boldsymbol{k}_{\rm t})B_{nj}^{-\dagger}(\boldsymbol{k}_{\rm t}) - \alpha_j^{-}(\boldsymbol{k}_{\rm t})B_{nj}^{+\dagger}(\boldsymbol{k}_{\rm t})\right)\frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2} \tag{16}$$

Insert the formulas of $\alpha_j^{\pm}(\boldsymbol{k}_{\rm t})$ from above in the expression for γ_n in (16), and we obtain an infinite set of equations that can be solved for every specified incident field. We write as

$$c_n = d_n + \sum_{n'n''}A_{nn'}T_{n'n''}c_{n''} \tag{17}$$

where the array $c_n=a_n+\gamma_n$, and the d_n vector is defined as

$$\begin{aligned} d_n &= -\sum_{j=1,2}\iint\limits_{\mathbb{R}^2}a_j^{+}(\boldsymbol{k}_{\rm t}) \\ &\times\frac{B_{nj}^{+\dagger}(\boldsymbol{k}_{\rm t})+(-1)^j{\rm e}^{2{\rm i}k_zz_-}B_{nj}^{-\dagger}(\boldsymbol{k}_{\rm t})}{1-{\rm e}^{-2{\rm i}k_zd}}\,\frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2} \\ &-\sum_{j=1,2}\iint\limits_{\mathbb{R}^2}a_j^{-}(\boldsymbol{k}_{\rm t}) \\ &\times\frac{B_{nj}^{-\dagger}(\boldsymbol{k}_{\rm t})+(-1)^j{\rm e}^{-2{\rm i}k_zz_+}B_{nj}^{+\dagger}(\boldsymbol{k}_{\rm t})}{1-{\rm e}^{-2{\rm i}k_zd}}\,\frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2}+a_n \end{aligned} \tag{18}$$

and the A matrix is

$$\begin{aligned} A_{nn'} &= -2\sum_{j=1,2}\iint\limits_{\mathbb{R}^2}B_{nj}^{-\dagger}(\boldsymbol{k}_{\rm t}) \\ &\times\frac{B_{n'j}^{-}(\boldsymbol{k}_{\rm t})+(-1)^j{\rm e}^{2{\rm i}k_zz_-}B_{n'j}^{+}(\boldsymbol{k}_{\rm t})}{1-{\rm e}^{-2{\rm i}k_zd}}\,\frac{k_0}{k_z}\,\frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2} \\ &-2\sum_{j=1,2}\iint\limits_{\mathbb{R}^2}B_{nj}^{+\dagger}(\boldsymbol{k}_{\rm t}) \\ &\times\frac{B_{n'j}^{+}(\boldsymbol{k}_{\rm t})+(-1)^j{\rm e}^{-2{\rm i}k_zz_+}B_{n'j}^{-}(\boldsymbol{k}_{\rm t})}{1-{\rm e}^{-2{\rm i}k_zd}}\,\frac{k_0}{k_z}\,\frac{{\rm d}k_x\,{\rm d}k_y}{k_0^2} \end{aligned} \tag{19}$$

The A matrix is independent of the excitation and the scatterer, and the entries can be computed once and for all and stored for later use.

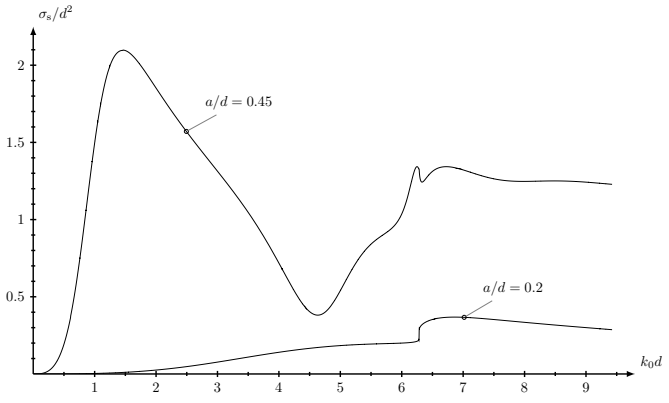

Fig. 2. The scattering cross section, $\sigma_{\rm s}$, for a PEC sphere of radius a as a function of $k_0 d$ and incident vertically polarized plane wave. The cross section is scaled with d^2. The location of the planes are $z_\pm/d = \pm 0.5$.

Finally, we get the expansion coefficients of the scattered field as, see (14)

$$\begin{cases} f_n = \displaystyle\sum_{n'} T_{nn'} c_{n'} \\ f_j^{\pm}(\boldsymbol{k}_{\rm t}) = \mp \dfrac{1}{2{\rm i}} \alpha_j^{\mp}(\boldsymbol{k}_{\rm t}) \end{cases} \tag{20}$$

VII. The primary and secondary fields

The total electric field $\boldsymbol{E}(\boldsymbol{r})$ between the plates can be decomposed in several ways. Above, we decomposed the total field as $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{\rm i}(\boldsymbol{r}) + \boldsymbol{E}_{\rm s}(\boldsymbol{r})$. As an alternative to this decomposition of the electric field, we introduce a decomposition in terms of a primary and secondary field, *i.e.*,

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{\rm i}(\boldsymbol{r}) + \boldsymbol{E}_{\rm s}(\boldsymbol{r}) = \boldsymbol{E}^{\rm prim}(\boldsymbol{r}) + \boldsymbol{E}^{\rm sec}(\boldsymbol{r})$$

The explicit expression of these fields are found in Ref. [9]. The field $\boldsymbol{E}^{\rm prim}(\boldsymbol{r})$ is the total electric field in the absence of the scatterer (surfaces S_+ and S_- present), and $\boldsymbol{E}^{\rm sec}(\boldsymbol{r})$ is the correction due to the presence of the scatterer.

All integrals that appear in the calculations can be evaluated exactly by calculus of residues [9], except for the entries of the A matrix, which can be evaluated exactly in terms of the Lerch function [11]

$$\Phi(\beta, \nu, \mu) = \sum_{n=0}^{\infty} \frac{\beta^n}{(\mu + n)^\nu}$$

VIII. Numerical examples

The first propagating mode is taken as the exciting field, which is a vertically polarized plane wave. In Figure 2 the scattering cross section for a perfectly conducting sphere of radius a is depicted as a function of $k_0 d$. The explicit data of the example are given in the caption.

IX. Conclusions

By the use of the integral representation of the scattered field, the solution to the complex electromagnetic scattering problem has been solved. The solution is an extension of the null field method, originally proposed by Peter Waterman, to geometries with two planar interfaces. Similar geometries have been addressed in the past, see *e.g.*, [5], but the present problem shows more complexity. The approach is well suited to numerical implementation, and a numerical example shows the usefulness of the method. Several ways of testing the numerical are available, *e.g.*, the optical theorem (appropriately formulated) and the sum rule. All these verification show excellent agreement.

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