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Vol. 1 pp. 243-246
Publication Date: 2014/03/17
Online ISSN: 2188-5079
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Abstract—Finite realizations of archetypal complex networks (including, most notably, the preferential attachment model) are typically mildly disassortative — in part due to the attachment of the most recent nodes, with fewest links, directly to the hubs. In contrast, the extant collection of real world networks exhibits a very wide range of assortativity. In this paper we explore both this bias and the opposite end of the spectrum: what can be done to make strongly assortative networks, what do they look like and, what effect does this have on the dynamical behaviour of such systems.

1. Introduction

Complex networks, and in particular, scale-free networks are widely described as ubiquitous throughout Nature. While this claim is reasonable, the widely utilized model for generating a scale-free network, preferential attachment [1], does introduce slight statistical bias for finite size realisations [6]. For a finite network, constructed via preferential attachment, the connections from the last added nodes are biased towards the hubs, and yet these nodes themselves have exceptionally low degree. This leads to finite scale free networks constructed with this method exhibiting negative assortativity. To overcome this we have proposed a form of altruistic attachment as an alternative: rather than connect directly to the hubs, new nodes are connected to random neighbors of the hubs. By doing this, the disassortativity of preferential attachment is entirely mitigated [3].

Here, we focus on the effect of this biased assortativity — or disassortativity — and ask: how does the structural and dynamical behaviour of a strongly assortative (or disassortative) network differ from the archetypal preferential attachment model? To quantify the structural properties of a network we measure an ensemble of the usual suspects: node degree, diameter, assortativity, clustering and robustness (to be briefly described latter). To measure dynamical behaviour we compute the leading eigenvalues of the fixed point equations of the network dynamical system. The measures of network structure which we employ are described below) to incrementally alter the assortativity.

In the following two sections we introduce the necessary machinery and then describe our computational results.

2. Networks and numerics

In the following three subsection we introduce the necessary numerical techniques: statistical quantification of network structure (Sec. 2.1), numerical measures of dynamical stability (Sec. 2.2), and methods of link manipulation to refine network assortativity (Sec. 2.3). In all of what follows we consider unweighted undirected graphs of $N$ nodes represented by an $N \times N$ symmetric adjacency matrix $A$ such that $A_{ij} = 1$ indicates the presence of a link between node $i$ and $j$ ($A_{ij} = 0$ otherwise). The main diagonal is zero ($A_{ii} = 0$).

2.1. Structure

The measures of network structure which we employ are described here briefly and refer the reader to [4] or the relevant sources for details.

- node degree: the average number of links for a node.
- diameter: the median over all $i$ and $j$ of the shortest path between node-$i$ and node-$j$.
- assortativity: the correlation coefficient between the degree of node-$i$ and the degree the neighbors of node-$i$ — computed over all $i$.
- clustering: the probability that two neighbors of node-$i$ are also neighbours — i.e. the prevalence of triangles within the network.
- robustness: the tendency for one of the properties of a network (usually diameter) to changes with targeted or random deletion of nodes — targeted deletion occurs when nodes which are deemed to contribute most to that property (high degree nodes, for the case of diameter) are removed first.

- 1 Or, nodes with highest betweenness — defined to be the fraction of shortest paths passing through that node. However, this is more costly to calculate.
2.2. Dynamics

Let \( \phi(x) \) define a dynamical system: \( x' = \phi(x) \). In what follows we restrict our attention to the one dimensional case and consider a concrete example for the purposes of simulation: \( \phi(x) = -x \). Clearly, this system is globally stable with a single stable node \( x_0 = 0 \). Define the following network dynamical system:

\[
    z' = (I + \epsilon A) \Phi(z)
\]

where \( I \) is the \( N \times N \) identity and \( \epsilon > 0 \) is a small positive coupling strength. The variable \( z \) consists of \( N \) sets of state variable of the system \( x' = \phi(x) \) (since, here, \( x \in \mathbb{R}, z \in \mathbb{R}^N \)) and \( \Phi(z) = [\phi(z_1) \cdots \phi(z_N)]' \) is \( N \) copies of the function \( \phi(z) \).

If \( A \) is full rank, then the solutions of (1) are \( z_0 \in \{z|\phi(z) = z(0) = 0\} \) where \( z^{(i)}(t) \) is the \( i-th \) component (or, in general, \( i-th \) subsystem) of \( z \). In the example we’ve chosen, this is trivial: \( z = 0 \). The stability of this single \( N \)-dimensional fixed point is determined via the eigenvalues of the matrix \( (I + \epsilon A) \Phi'(z_0) \). For \( \epsilon \ll 1 \) these eigenvalues are all \(-1\). Hence, we are interested only in the largest eigenvalue of \( (I + \epsilon A) \) and the value of \( \epsilon \) of which is becomes positive — at this point the system gains a single unstable direction, becomes a saddle, and becomes unbounded.

2.3. Links

The four networks identified in the introduction exhibit a fairly wide range of assortativity — two of them (the skinny network and the small-world network) are highly assortative. Nonetheless, how much more assortative (or disassortative) can a network be? To explore a wider range of network assortativity we implement the following simple link exchange mechanism:\footnote{Here, node means a fixed point of a dynamical system with real eigenvalues — nothing to do with the network.}

1. pick two nodes \( i \) and \( j \) at random such that their degrees \( d(i) \) and \( d(j) \) differ
2. pick neighbors of these nodes \( \hat{i} \) and \( \hat{j} \) such that the metrics
   \[ |d(\hat{i}) - d(j)| \]
   and
   \[ |d(\hat{j}) - d(i)| \]
   are minimised
3. exchange links: disconnect node-\( \hat{i} \) from node-\( i \) and instead connect it to node-\( j \), similarly disconnect node-\( \hat{j} \) from node-\( j \) and instead connect it to node-\( i \)
4. repeat

Note that the individual degrees of each node (and hence the degree distribution) are preserved, but since the degrees of the original nodes differ \( d(i) \neq d(j) \) and link exchange mechanism attempts to produce strong links between nodes with similar degrees, the resultant network will tend to have a higher assortativity. Conversely, to decrease assortativity we do the reverse, step 2 is replaced with the following:

2’ pick neighbors of these nodes \( \hat{i} \) and \( \hat{j} \) such that the metrics
   \[ |d(\hat{i}) - d(i)| \]
   and
   \[ |d(\hat{j}) - d(j)| \]
   are minimised.

Figure 1: Varying assortativity with a link-exchange mechanism: The six panels depict computation of networks properties described in the text after repeated application of link-exchange to either increase or decrease the assortativity. The four initial networks are: BA preferential attachment (red); the WS small-world model (green); “altruistic attachment” (blue); and, the “skinny” network (black). Mean and standard deviations from 52 trials are depicted. The horizontal axis (for every panel) is the extent to which the link-exchange mechanism was applied to increase (positive) or decrease (negative) the assortativity. Link-exchange was applied between 100 and 50000 times.
That is, instead of seeking to achieve a good match, we seek to destroy an existing good match. Over time, these iterative schemes can be applied to significantly increase or decrease network assortativity. In the next section we present the results of our computational study of these networks.

3. Calculation

We have performed an extensive set of simulations to fully explore the parameter space delineated in the previous section. Figure 1 highlights some of these results. For each of the four networks described in Sec. 1 we repeatedly apply the link-exchange mechanisms of Sec. 2.3 to increase and decrease the network assortativity. For each network, we then compute the range of statistics described in Sec. 2.1 and 2.2.

From Fig. 1 we can draw several results. The link-exchange mechanism does have the desired effect on assortativity and has little effect on scale-free exponent (for networks from which this could be reliably estimated). Network degree is invariant — as required by the link-exchange mechanism. The link modification seems to have a symmetric effect on both path-length and clustering of the small-world and skinny networks. Unlike the preferential and altruistic attachment models, both these networks are embeddable, or almost embeddable in two dimensions (such that nodes that are close according to a Euclidean metric are also connected). The link modification scheme disrupts this property and hence decreases clustering and path-length — as the networks become more random.

Finally, we observe that decreasing assortativity causes a nonlinear increase in the maximum eigenvalue of the resultant network for the scale-free networks with some notable exceptions. More assortative networks are more stable: and the effect is most pronounced for the most extremely assortative or disassortative networks. In Fig. 2 and Fig. 3 we probe this relationship further and ask what can one learn about the dynamical structure of the network from examining the structural properties of path-length and assortativity. Of course, with all else being equal a more assortative network will typically have a larger diameter (path-length) as high degree nodes will tend to be wired together, rather than being connected to different regions of the network.

In Fig. 2 we see the effect of assortativity more clearly — increasing assortativity decreases the maximum eigenvalue (making the network dynamically more stable). While widely distributed, one does observe that for a given value of assortativity the maximum eigenvalues is bounded above by a quantity which decreases linearly with assortativity. In Fig. 3 we see a similar relationship between mean path length and maximum eigenvalue. Taking all three quantities together we find that the maximum eigenvalue can be predicted from a combination of assortativity and path-length — in three dimensions the set of points depicted here approximates a smooth two-dimensional surface. Figure 4 depicts exactly this surface. We build a model, using methods we developed previously[2, 5], to predict stability (as defined above) from just assortativity and path-length. The error in these predictions (when applied to new data) has a mean of 0.027 (about 15%).

Finally, in Fig. 5 we compute the effect of changing the network assortativity on the robustness of the network. As shown in the upper panel, for all networks, targeted removal results in a linear increase in network size as the removal fraction increases — up to the point where the network fractures. Random removal has very little effect on network size. This effect is robust across all levels of initial assortativity. Hence, the robust and fragile property of these networks is preserved. Conversely, targeted node removal has a very marked effect on network assortativity — by removing a small fraction of nodes assortativity is very quickly attenuated.
4. Conclusions

It is not sufficient to collectivise all scale free networks and treat them equally. While some properties — such as the so-called “robust-yet-fragile” character of these networks is preserved upon manipulation of assortativity — this is not always the case. Other properties are sensitively dependent on the local connection between nodes. In particular, more assortative networks are more dynamically stable. By merely changing the assortativity on a network one can progress from a regime of stable equilibrium dynamics to global instability. Nonetheless, we do find that, to a very large degree, stability of a network can be predicted from just network diameter and path-length — two global properties of connectivity that are, in turn, strongly affected by assortativity.

References