

Time-Domain Analysis of Plane Wave Reflection from a Dispersive Dielectric Half-Space

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Abstract— In this paper, the time-domain pulse reflection from a dispersive dielectric half-space is investigated. The properties of a half-space are described in the frequency-domain by the Debye model, which is commonly used to capture the relaxation-based dispersive properties. Firstly, transient reflected pulses are analyzed using a time-domain convolution. Then, based on the analysis, the waveform parameters of reflected pulses are estimated, and the relationships between the waveform parameters and the properties of dispersive material and incident angles are discussed. Excellent agreement is obtained between our results and those in the literature, supporting the correctness and effectiveness of this work.

Key words: transient analysis, pulse reflection, dispersive dielectric half-space, time-domain convolution, waveform parameter estimation.

I. INTRODUCTION

The reflection of short duration electromagnetic pulses from dielectric media is of interest in the fields of electromagnetic compatibility (EMC) and antennas. Many kinds of media show relaxation-based dispersive properties. The two models commonly used to capture relaxation-based dispersive properties are the Debye [1] and Cole–Cole [2] models. For many types of materials including biological tissues, Cole–Cole models provide an excellent fit to experimental data over the entire measurement frequency range. However, because of the computational complexity of embedding a Cole–Cole dispersion model into numerical methods, a variety of Debye models that achieved both simplicity and accuracy were developed and applied. So far, the simpler one- and two-pole Debye equations with loss terms have been used to describe dielectric properties of building materials [3], circuit boards [4], ceramics [5] and biological tissues [6] [7].

A typical approach to material characterization is to examine reflected electromagnetic pulses from the interface between free-space and the investigated material. It is therefore necessary to develop an efficient technique to analyze the transient reflection from a Debye material. Rothwell [8] worked out the time-domain reflection coefficients of a Debye half-space for both horizontal and vertical polarizations that involve exponential and modified Bessel functions and require convolution operations to evaluate. Very recently, we have developed an efficient method on the basis of the numerical inversion of the Laplace transform for calculation of time-domain surface impedance

of a lossy half-space [9]. It is the purpose of this paper to develop a new technique for solving the impulse response of a Debye half-space without performing tedious and complicated mathematical manipulations. This technique leads to good accuracy, and has a simple algorithm, short calculation time, small required memory size, readily controlled error and wide range of applicability. The numerical results show excellent agreement with those in [8], validating the effectiveness of this technique.

II. THEORY ON NUMERICAL INVERSION OF LAPLACE TRANSFORM

The Laplace transform (image function in complex frequency-domain) $F(s)$ and the inverse Laplace transform (original function in time-domain) $f(t)$ are related by the forward transformation

$$Lf(t) = F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (1)$$

and the inverse transformation

$$L^{-1}F(s) = f(t) = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} F(s)e^{st} ds. \quad (2)$$

In general, it is straightforward to take the Laplace transform of a function. However, the inverse transformation is often difficult. In many cases, the method using simple rules and a table of transforms, and the method using the Bromwich integral and Cauchy integral theorem do not work well. A numerical technique has then to be utilized. In this work, we apply the method proposed by Hosono [10]. To apply his numerical inversion method, the following conditions must be satisfied.

- 1) $F(s)$ is defined for $\text{Re}(s) > 0$;
- 2) $F(s)$ is nonsingular;
- 3) $\lim_{s \rightarrow \infty} F(s) = 0$ for $\text{Re}(s) > 0$;

The most distinctive feature of this method lies in the approximation of e^{st} . The main points are listed as follows.

$$\text{i) } e^{st} = \lim_{\rho \rightarrow \infty} \frac{e^{\rho}}{2 \cosh(\rho - st)}. \quad (3)$$

$$\text{ii) } e^{st} \approx E_{ec}(st, \rho) = \frac{e^{\rho}}{2 \cosh(\rho - st)} \quad (4)$$

$$= e^{st} - e^{-2\rho} e^{3st} + e^{-4\rho} e^{5st} - \dots \quad (5)$$

$$= \frac{e^\rho}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n j}{st - [\rho + j(n-0.5)\pi]} \quad (6)$$

iii) The Bromwich integral is transformed to the integral around the poles of $E_{ec}(st, \rho)$.

Then $f(t)$ is approximated by $f_{ec}(t, \rho)$, defined by

$$f_{ec}(t, \rho) = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} F(s) E_{ec}(st, \rho) ds \quad (7)$$

$$= f(t) - e^{-2\rho} f(3t) + e^{-4\rho} f(5t) - \dots \quad (8)$$

$$= (e^\rho/t) \sum_{n=1}^{\infty} F_n \quad (9)$$

where $t > 0$, and

$$F_n = (-1)^n \text{Im} F \left\{ \left[\rho + j(n-0.5)\pi \right] / t \right\}. \quad (10)$$

Equation (8) shows that the function $f_{ec}(t, \rho)$ gives a good approximation to $f(t)$ when $\rho \gg 1$ and can be used for error estimation. Equation (9) is derived by substituting (6) into (7), and may be used for the effective numerical inversion of the Laplace transform. In practice, the infinite series in (9) has to be truncated after an appropriate number of terms. Since the infinite series is a slowly convergent alternating series, truncating to a small number of terms leads to a significant error. An effective approach using the Euler transformation has been developed, under the following conditions [11]:

a) There exists an integer $k \geq 1$ such that the signs of F_n alternate for $n \geq k$;

b) For $n \geq k$, $\frac{1}{2} < |F_{n+1}/F_n| \leq 1$.

With conditions a) and b) met, (9) can be truncated to $f_{ec}^{lm}(t, a)$, which has $N = l + m$ terms and is given by

$$f_{ec}^{lm}(t, \rho) = (e^\rho/t) \left(\sum_{n=1}^{l-1} F_n + 2^{-m-1} \sum_{n=0}^m A_{mn} F_{l+n} \right) \quad (11)$$

where A_{mn} are defined recursively by

$$A_{mm} = 1, \quad A_{mn-1} = A_{mn} + \binom{m+1}{n}. \quad (12)$$

In this method, the upper bound of the truncation error is given by

$$R^{lm} = \left| f_{ec}^{l+1,m}(t, \rho) - f_{ec}^{l,m}(t, \rho) \right| \quad (13)$$

while the upper bound of the approximation error is given by

$$\left| f_{ec}(t, \rho) - f(t) \right| \approx M e^{-2\rho}, \quad (14)$$

if

$$\left| f(t) \right| \leq M \quad \text{for all } t > 0.$$

As indicated in (14), the relative approximation error is less than $e^{-2\rho}$, while the truncation error increases with t and

decreases with N . For a typical value of t , the calculation is repeated by increasing N by 5 for an estimation of the truncation error.

III. TIME-DOMAIN REFLECTION COEFFICIENTS

Without losing generality and for the comparison with results in [8], a one-pole Debye model with zero loss is utilized in this work. The Laplace variable $s = j\omega$ is introduced, and an interface between free-space and a dielectric half-space with unity permeability and a permittivity $\varepsilon(s) = \varepsilon_0 \varepsilon_r(s)$ is considered, described by the Debye equation

$$\varepsilon_r(s) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + s\tau}, \quad (15)$$

where ε_s is the static permittivity, ε_∞ is the optical permittivity, $\varepsilon_s > \varepsilon_\infty$, and τ is the relaxation time.

A plane wave is obliquely incident onto the Debye half-space from free-space, at an incidence angle θ relative to the normal to the interface. For horizontal polarization, the reflection coefficient is given by

$$R_H(s) = \frac{\cos \theta - \sqrt{\varepsilon_r(s) - \sin^2 \theta}}{\cos \theta + \sqrt{\varepsilon_r(s) - \sin^2 \theta}}. \quad (16)$$

Substituting for $\varepsilon_r(s)$ from (15) leads to

$$R_H(s) = \frac{\sqrt{s+s_0} - K_H \sqrt{s+s_1}}{\sqrt{s+s_0} + K_H \sqrt{s+s_1}}, \quad (17)$$

where

$$s_0 = \frac{1}{\tau}, \quad s_1 = \frac{1}{\tau} \frac{\varepsilon_s - \sin^2 \theta}{\varepsilon_\infty - \sin^2 \theta} > s_0, \quad (18)$$

and

$$K_H = \frac{\sqrt{\varepsilon_\infty - \sin^2 \theta}}{\cos \theta}. \quad (19)$$

For vertical polarization, the reflection coefficient is given by

$$R_V(s) = \frac{\sqrt{\varepsilon_r(s) - \sin^2 \theta} - \varepsilon_r(s) \cos \theta}{\sqrt{\varepsilon_r(s) - \sin^2 \theta} + \varepsilon_r(s) \cos \theta}. \quad (20)$$

Substituting $\varepsilon_r(s)$ from (15) leads to

$$R_V(s) = \frac{\sqrt{s+s_0} \sqrt{s+s_1} - K_V (s+s_2)}{\sqrt{s+s_0} \sqrt{s+s_1} + K_V (s+s_2)}, \quad (21)$$

where

$$s_2 = \frac{1}{\tau} \frac{\epsilon_s}{\epsilon_\infty}, \quad (22)$$

and

$$K_V = \frac{\epsilon_\infty \cos \theta}{\sqrt{\epsilon_\infty - \sin^2 \theta}}. \quad (23)$$

It is noted that either $R_H(s)$ or $R_V(s)$ does not satisfy condition 3) of Section II, that is, is not asymptotic to zero at high frequency, but instead

$$\lim_{s \rightarrow \infty} R_H(s) = R_H^\infty = \frac{1 - K_H}{1 + K_H} \quad (24)$$

and

$$\lim_{s \rightarrow \infty} R_V(s) = R_V^\infty = \frac{1 - K_V}{1 + K_V}. \quad (25)$$

Subtracting terms R_H^∞ and R_V^∞ from $R_H(s)$ and $R_V(s)$ respectively gives the “reduced” reflection coefficients,

$$\bar{R}_H(s) = R_H(s) - R_H^\infty = \frac{2K_H}{1 + K_H} \frac{\sqrt{s + s_0} - \sqrt{s + s_1}}{\sqrt{s + s_0} + K_H \sqrt{s + s_1}}, \quad (26)$$

and

$$\bar{R}_V(s) = R_V(s) - R_V^\infty = \frac{2K_V}{1 + K_V} \frac{\sqrt{s + s_0} \sqrt{s + s_1} - (s + s_2)}{\sqrt{s + s_0} \sqrt{s + s_1} + K_V (s + s_2)}. \quad (27)$$

Both $\bar{R}_H(s)$ and $\bar{R}_V(s)$ satisfy conditions 1) – 3) in Section II, under which Equation (9) holds. It can be proved that, for $s = [\rho + j(n - 0.5)\pi] / t$, both $\bar{R}_H(s)$ and $\bar{R}_V(s)$ also obey the two conditions a) and b) in Section II, under which $f_{ec}^{lm}(t, \rho)$ can be used to approximate $f_{ec}(t, \rho)$ (proof omitted because of space limitations). Hence, both reduced time-domain reflection coefficients $\bar{R}_H(t)$ and $\bar{R}_V(t)$ can be calculated using Equation (11). The required time-domain reflection coefficients $R_H(t)$ and $R_V(t)$ are then obtained by adding $R_H^\infty(t)$ and $R_V^\infty(t)$ to $\bar{R}_H(t)$ and $\bar{R}_V(t)$ respectively.

Before applying this technique to waveform parameter estimation, its correctness and effectiveness are verified by comparing the reduced transient reflection coefficients with those in [8]. Consider a water half-space and assume that the permittivity of water follows the Debye model. Figure 1 illustrates the reduced reflection coefficients of water (at standard temperature and pressure) calculated using our technique, and shows excellent agreement with [8]. The reduced reflection coefficients do not include any impulsive component with amplitude of R_H^∞ or R_V^∞ . The large scale on the vertical axis may be disconcerting at first look, but it should be noted that these reflection coefficients will be convolved with incident pulses with durations on the order of nanoseconds. Consider a Gaussian waveform incident upon a water half-space at $\theta = 30^\circ$. The permittivity of water is

assumed to follow the Debye model. The incident field is horizontally polarized and has an amplitude of 1 V/m and a pulse width of 1 ps. The reflected waveform can be determined using the convolution,

$$E_H^r(t) = R_H(t) * E^i(t) = \bar{R}_H(t) * E^i(t) + R_H^\infty E^i(t),$$

where $\bar{R}_H(t)$ is shown in Figure 1 and R_H^∞ is given by (24). The reflected waveform is plotted in Figure 2, and shows excellent agreement with Figure 4 in [8]. Also in Figure 2, it is seen that the incident Gaussian waveform is maintained, but with a long tail contributed by the waveform of $\bar{R}_H(t)$ due to the relaxation effect.

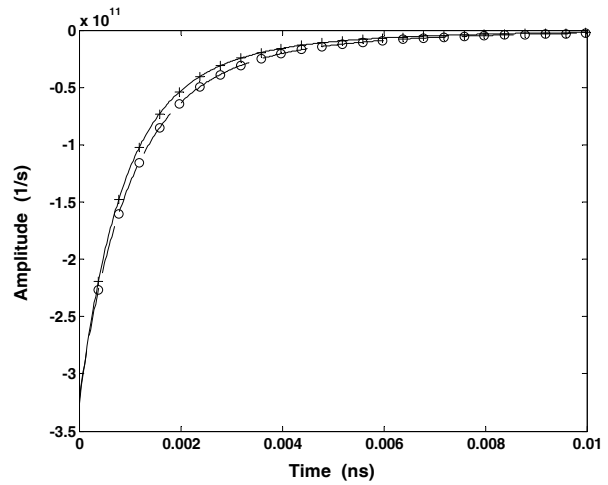


Fig. 1 Time-domain reduced reflection coefficients for water, $\epsilon_s = 78.3$, $\epsilon_\infty = 5.0$, $\tau = 9.6 \times 10^{-12}$ s, and $\theta = 30^\circ$. Solid line: Our results for horizontal polarization; Plus sign: Results for horizontal polarization in [8]; Dashed line: Our results for vertical polarization; Circle: Results for vertical polarization in [8].

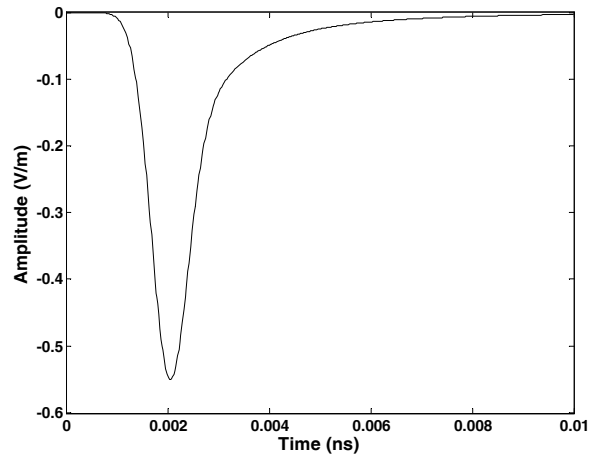


Fig. 2 Reflected waveform for a horizontally polarized Gaussian pulse incident on a water half-space at $\theta = 30^\circ$.

IV. WAVEFORM PARAMETER ESTIMATION

Based on the above transient analysis, this technique can be utilized for the estimation of waveform parameters of reflected pulses. As an example, consider a mixture of water and ethanol with a volume fraction v_F . Here, $v_F = 0$ corresponds to pure ethanol while $v_F = 1$ corresponds to pure water. Bao et al. have shown the permittivity of this mixture is described quite well by the Debye model and have measured the Debye parameters for various volume fractions. The parameters can be approximated by the following expressions [12]:

$$\varepsilon_\infty = -19.1 v_F^2 + 18.5 v_F + 4.8, \quad (28)$$

$$\varepsilon_s - \varepsilon_\infty = 53 v_F + 22, \quad (29)$$

$$\tau = 0.15 \times 10^{-1.27 v_F} \text{ ns}. \quad (30)$$

One of the most important waveform parameters is the correlation between two waveforms. It indicates the degree to which two waveforms resemble each other and is defined by

$$C(t) = \left(\frac{\int_0^\infty s_1(t') s_2(t+t') dt'}{s_m} \right)^2 \quad (31)$$

$$s_m = \max \left(\int_0^\infty s_1^2(t') dt', \int_0^\infty s_2^2(t') dt' \right) \quad (32)$$

Let $s_1(t)$ and $s_2(t)$ be the incident and reflected waveforms, respectively, and consider the Gaussian waveform in Section III incident upon a mixture half-space. The maximum value C_{\max} of $C(t)$ is plotted versus the volume fraction v_F for three incident angles in Figure 3. C_{\max} is seen to increase with v_F and θ .

V. CONCLUSIONS

A new technique has been developed for the transient analysis of pulse reflections from a Debye medium. Transient reflected pulses were analyzed using a time-domain convolution. Then, based on the analysis, the waveform parameters of reflected pulses were estimated, and the relationships between the waveform parameters of reflected pulses and the properties of dispersive material and incident angles discussed. The numerical results show excellent agreement with those in [8], supporting the effectiveness of this technique. This technique is based on the numerical inversion of the Laplace transform and has several significant advantages.

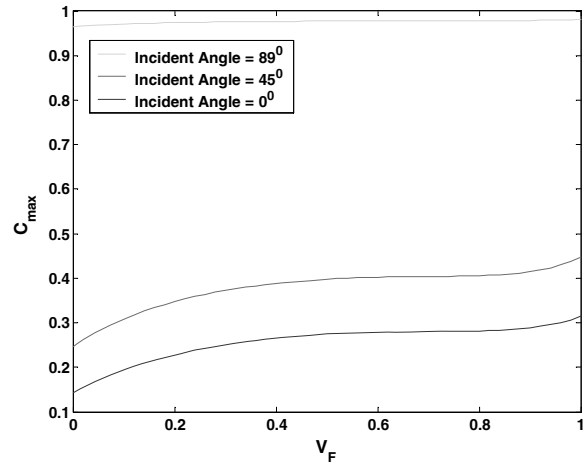


Fig. 3 Maximum correlation between the incident and reflected waveforms for a mixture irradiated by a horizontally polarized Gaussian pulse.

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