

High Frequency Diffraction by Strongly Elongated Spheroids

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Abstract—Diffraction of high-frequency electromagnetic wave by strongly elongated spheroids is studied. Previous results are generalized to the case of skew incidence.

I. INTRODUCTION

The rate of elongation of a body significantly affects the diffraction phenomena. As the result high-frequency asymptotics which do not take into account the elongation may be not accurate and even give a wrong approximations for the electromagnetic field. Assuming that the radii of curvature ρ and ρ_t in longitudinal and transverse cross-sections of the body satisfy the relation

$$k^2 \rho_t^3 / \rho \equiv \chi = O(1), \quad (1)$$

where k is the wave number, we derive a set of special asymptotic expansions which possess uniformity with respect to the rate of elongation. The high-frequency diffraction by a strongly elongated spheroid was studied in [1], [2]. In the leading order the current induced by an incident plane is given by the asymptotics

$$J = \exp(ikz) A\left(\frac{z}{\sqrt{\rho\rho_t}}, \chi\right) \sin \varphi, \quad (2)$$

where A is a special function defined by the integral

$$A(\eta, \chi) = \frac{2 \exp(i\pi/4) \exp(-i\chi\eta/2)}{\pi \sqrt{\chi} \sqrt{1-\eta^2}} \int_{-\infty}^{+\infty} \left(\frac{1-\eta}{1+\eta}\right)^{it} \times \frac{\Gamma(1/2 + it) W_{it,1}(-i\chi) dt}{W_{it,0}(-i\chi) W_{it,1}(-i\chi) + W_{it,0}(-i\chi) W_{it,1}(-i\chi)} \quad (3)$$

with M and W being Whittaker functions.

When $\chi \rightarrow \infty$, that is when the body becomes not too much elongated, asymptotics (2) transforms to the usual Fock formula. Comparison with test computations [3] shows that formulae (2), (3) give a very accurate approximation of the current in a wide range of values of elongation parameter χ (see Fig. 1).

Further development of the approach in [4] was in the analysis of the wave that reflects from the shaded end of the spheroid and propagates in backward direction. It asymptotics is given by the formula similar to (2), but with an additional multiplier under the integral in (3). Interference of forward and backward waves form oscillations seen on numerical curves on Fig. 1.

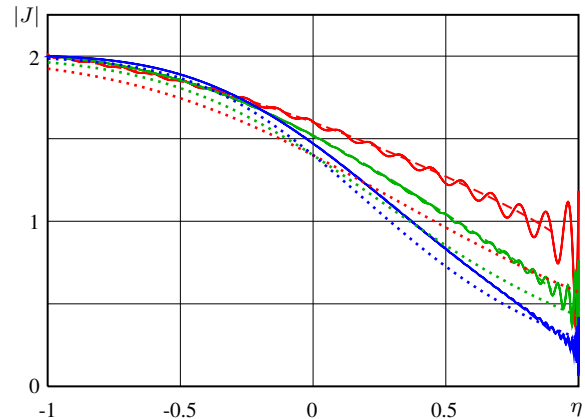


Fig. 1. Induced currents on the spheroid with semi-axes $a = 1$ m, $b = 2.5$ m at frequencies 0.5 GHz (red), 1 GHz (green) and 2 GHz (blue). Elongation parameters χ are 4.19, 8.38 and 16.77 correspondingly to 0.5, 1 and 2 GHz. Solid curves present numerical solution, dashed — asymptotics (2), dotted — Fock asymptotics.

All these results were obtained for the axially symmetric problems, that is the incident plane wave was running along the spheroid axis. In this paper we generalized the above results to the case of incidence at a small angle to the axis.

II. ASYMPTOTIC PROCEDURE

Assumption (1) means that the semi-axes of the spheroid, large b and small a , are such that $ka^2/b \equiv \chi = O(1)$. The asymptotic procedure follows the usual steps. First, the boundary layer of thickness having the order $O((k\rho)^{-2/3})$ and length of the order $O((k\rho)^{-1/3})$ is introduced. The boundary layer has the usual scale, but the whole spheroid lies in that layer. This brings to the necessity to introduce special coordinates. We introduce η, τ by the formulae

$$z = b\eta + \frac{\chi}{2k}(\tau - 1)\eta, \quad r = \sqrt{\frac{b\chi}{k}(1-\eta^2)}\tau.$$

Coordinate η is the angular spheroidal coordinate, it varies on the interval $[-1, 1]$, while τ is the scaled radial coordinate, $\tau > 0$ and $\tau = 1$ on the surface.

The electromagnetic field \mathbf{E} , \mathbf{H} is searched in the form of Fourier series by the angular coordinate φ . Maxwell equations allow all the components of the field to be expressed via E_φ and H_φ . After extracting the quick oscillating multiplier $\exp(ikb\eta)$ and neglecting smaller order terms in the equations

for E_φ and H_φ we get a system of "parabolic" equations. For new unknowns P_ℓ and Q_ℓ such that

$$\begin{aligned} E_\varphi &= e^{ikb\eta + i\ell\varphi} \{P_\ell(\eta, \tau) + Q_\ell(\eta_\tau)\}, \\ H_\varphi &= e^{ikb\eta + i\ell\varphi - i\pi/2} \{P_\ell(\eta, \tau) - Q_\ell(\eta_\tau)\} \end{aligned}$$

the system for each harmonics reduces to two independent equations

$$L_{\ell-1}P_\ell = 0, \quad L_{\ell+1}Q_\ell = 0, \quad (4)$$

where

$$\begin{aligned} L_n &= \tau \frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial \tau} + \frac{i\chi}{2} (1 - \eta^2) \frac{\partial}{\partial \eta} \\ &\quad + \left(\frac{\chi^2}{4} \tau - \frac{n^2}{4\tau} - \frac{\chi^2}{4} (1 - \eta^2) - \frac{i\chi}{2} \eta \right). \end{aligned}$$

Equations (4) are supplied with boundary conditions on the surface

$$\begin{aligned} P_\ell(\eta, 1) + Q_\ell(\eta, 1) &= 0, \\ \left(\frac{\partial}{\partial \tau} + \frac{1}{2} \right) (P_\ell(\eta, 1) - Q_\ell(\eta, 1)) &= 0 \end{aligned} \quad (5)$$

and radiation conditions at $\tau \rightarrow +\infty$.

By variables separation we solve equations (4), which allows P_ℓ and Q_ℓ to be represented in the following form

$$\begin{aligned} P_\ell &= \frac{e^{-i\chi\eta/2}}{\sqrt{1-\eta^2}\sqrt{\tau}} \int_{-\infty}^{+\infty} \left(\frac{1-\eta}{1+\eta} \right)^{it} \times \\ &\quad \times \left\{ p_\ell(t) M_{it, \frac{n-1}{2}}(-i\chi\tau) + p'_\ell(t) W_{it, \frac{n-1}{2}}(-i\chi\tau) \right\} dt, \end{aligned} \quad (6)$$

$$\begin{aligned} Q_\ell &= \frac{e^{-i\chi\eta/2}}{\sqrt{1-\eta^2}\sqrt{\tau}} \int_{-\infty}^{+\infty} \left(\frac{1-\eta}{1+\eta} \right)^{it} \times \\ &\quad \times \left\{ q_\ell(t) M_{it, \frac{n+1}{2}}(-i\chi\tau) + q'_\ell(t) W_{it, \frac{n+1}{2}}(-i\chi\tau) \right\} dt. \end{aligned} \quad (7)$$

The multipliers p_ℓ , p'_ℓ , q_ℓ and q'_ℓ are arbitrary at this step. The terms containing Whittaker functions M are regular on the axis of the spheroid and correspond to the incident wave, while terms with Whittaker function W satisfy the radiation condition and represent reflected wave.

Further we consider the incident plane wave. It can be represented as the sum of TE and TM waves

$$\begin{aligned} \mathbf{E}^{TE} &= \exp(ikz \cos \vartheta + ikx \sin \vartheta) \mathbf{e}_y, \\ \mathbf{H}^{TM} &= \exp(ikz \cos \vartheta + ikx \sin \vartheta) \mathbf{e}_y. \end{aligned}$$

In the case of small angle $\vartheta = O((kb)^{-1/2})$ these waves satisfy "parabolic" equations (4), so we equate these waves in a boundary layer to representations (6), (7). This yields a system of integral equations for the amplitudes p_ℓ and q_ℓ . These equations are of the type of convolution on the interval [5] and can be solved explicitly.

Finally, substitution of representations (6), (7) with already known amplitudes p_ℓ and q_ℓ into the boundary conditions (5) yields expressions for p'_ℓ and q'_ℓ .

Setting $\tau = 1$ in the formulae for H_φ we find the asymptotics of the longitudinal component of the induced current

$$J^{TE} = e^{ikb\eta} A^{TE}(\eta, \chi, \sqrt{kb}\vartheta), \quad (8)$$

$$J^{TM} = e^{ikb\eta} A^{TM}(\eta, \chi, \sqrt{kb}\vartheta). \quad (9)$$

The special functions A^{TE} and A^{TM} can be most compactly written in terms of Coulomb wave functions F_L and H_L^\pm

$$\begin{aligned} A^{TE}(\eta, \chi, \beta) &= -\frac{8}{\pi} \frac{e^{-i\chi\eta/2}}{\sqrt{1-\eta^2}\sqrt{\chi}\beta} \int_{-\infty}^{+\infty} \left(\frac{1-\eta}{1+\eta} \right)^{it} \times \\ &\quad \times \sum_{n=1}^{\infty} \frac{i^n \sin(n\varphi)}{Z_n(t)} \left[F_{\frac{n}{2}} \left(t, \frac{\beta^2}{2} \right) H_{\frac{n}{2}-1}^+ \left(-t, \frac{\chi}{2} \right) \right. \\ &\quad \left. + F_{\frac{n}{2}-1} \left(t, \frac{\beta^2}{2} \right) H_{\frac{n}{2}}^+ \left(-t, \frac{\chi}{2} \right) \right] dt, \end{aligned} \quad (10)$$

$$\begin{aligned} A^{TM}(\eta, \chi, \beta) &= \frac{8}{\pi} \frac{e^{-i\chi\eta/2}}{\sqrt{1-\eta^2}\sqrt{\chi}\beta} \int_{-\infty}^{+\infty} \left(\frac{1-\eta}{1+\eta} \right)^{it} \times \\ &\quad \times \left\{ \frac{F_0(t, \beta^2/2)}{H_0^+(-t, \chi/2)} \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{i^n \cos(n\varphi)}{Z_n(t)} \left[F_{\frac{n}{2}} \left(t, \frac{\beta^2}{2} \right) H_{\frac{n}{2}-1}^+ \left(-t, \frac{\chi}{2} \right) \right. \right. \\ &\quad \left. \left. - F_{\frac{n}{2}-1} \left(t, \frac{\beta^2}{2} \right) H_{\frac{n}{2}}^+ \left(-t, \frac{\chi}{2} \right) \right] \right\} dt, \end{aligned} \quad (11)$$

where

$$\begin{aligned} Z_n(t) &= H_{\frac{n}{2}-1}^+ \left(-t, \frac{\chi}{2} \right) \dot{H}_{\frac{n}{2}}^+ \left(-t, \frac{\chi}{2} \right) \\ &\quad + \dot{H}_{\frac{n}{2}-1}^+ \left(-t, \frac{\chi}{2} \right) H_{\frac{n}{2}}^+ \left(-t, \frac{\chi}{2} \right). \end{aligned}$$

For $\beta = 0$ functions A^{TE} and A^{TM} reduce to A from (3).

III. NUMERICAL RESULTS

For the computation of Coulomb wave functions we used FORTRAN program from [6]. Numerical analysis shows that subintegral expressions in (10), (11) exponentially decrease at $\pm\infty$. The number of terms in the series that are significant is not large. It grows with β , but for all the results presented in this paper it was less than 15.

Absolute values of the currents on the surface of spheroids those elongation parameters are $\chi = 10$ and $\chi = 1$ are presented on Figs. 2 and 3. For axially incident wave, the TE and TM polarizations do not differ except by the rotation by $\Delta\varphi = \pi/2$. On not to much elongated spheroid of Fig. 2 currents rapidly decrease with η , however this decrease is slower than predicted by Fock asymptotics. On more elongated spheroid of Fig. 3 the attenuation becomes slower.

Distributions of currents for skew incidence change and more noticeably for TM polarization. In the case of large χ (smaller elongation) or large β (larger angle of incidence) the character of currents distribution is more or less consistent

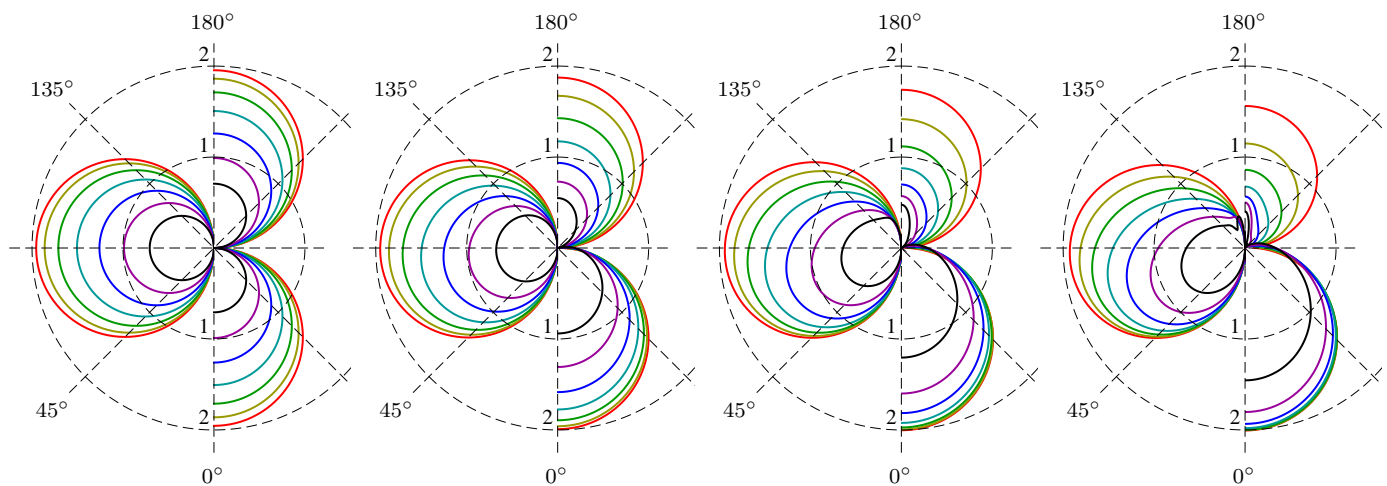


Fig. 2. Absolute values of currents on elongated spheroid for $\chi = 10$, at sections $\eta = -0.75, -0.5, -0.25, 0, 0.25, 0.5$ and 0.75 (colors from red to black), at angles of incidence $\beta = 0, 1, 2$ and 3 (from left to right), TE polarization on the left halves and TM polarization on the right halves.

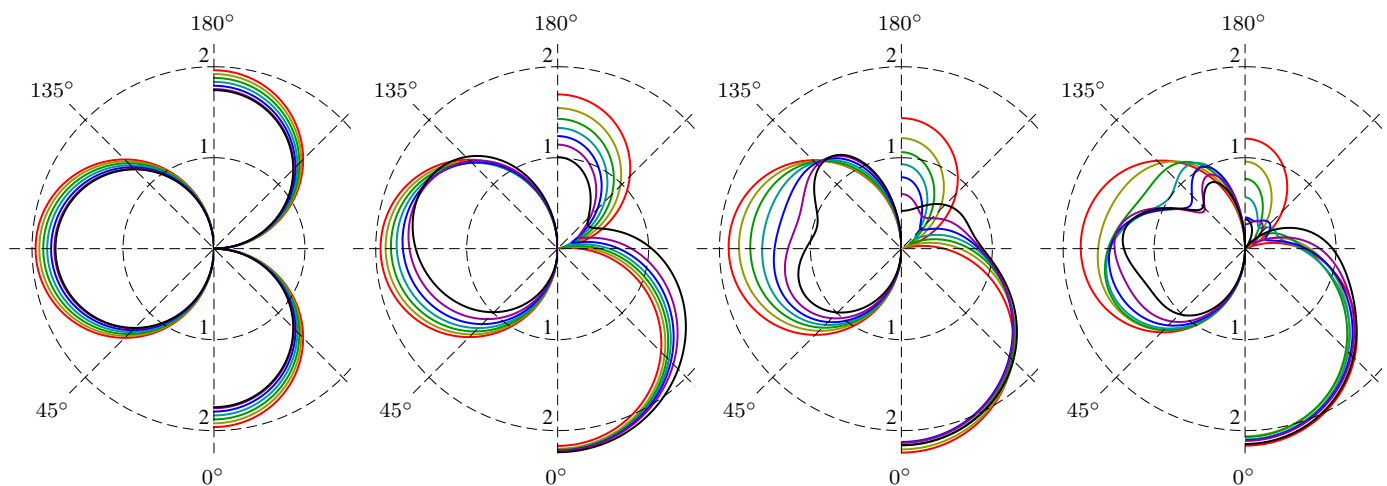


Fig. 3. Same as on Fig. 2, but for $\chi = 1$.

with the predictions of the usual high-frequency asymptotics. On the side that turns to light currents grow with β and on the opposite side except at the shadow ending of the spheroid they decrease. Near the shadowed ending there appear peaks of current when β is sufficiently large. These peaks are due to the interference of creeping waves that run from the illuminated side along the geodesics. The values of these peaks are larger on more elongated spheroid because creeping waves are less attenuated in this case.

One can be also notice that the effect of incidence angles corresponding to the same value of β is larger on more elongated spheroids. This can be explained by noting that the position of the light-shadow boundary on the surface is defined by the quantity $\alpha = \beta/\sqrt{\chi}$. At section $\varphi = 0^\circ$, for example, it is at $\eta = -\alpha/\sqrt{1+\alpha^2}$.

It is worth noting a specific effect for small χ and β . The current of TM wave increases on the illuminated side (at $|\varphi| < 90^\circ$), it becomes greater than 2 and grows with η .

REFERENCES

- [1] I. V. Andronov, "High-frequency asymptotics for diffraction by a strongly elongated body", *IEEE Antennas Wireless Propag. Lett.*, vol.8, p. 872, 2009.
- [2] I. V. Andronov, "High frequency asymptotics of electromagnetic field on a strongly elongated spheroid", *PIERS online*, vol. 5, No. 6, p. 536–540, 2009.
- [3] I. V. Andronov, D. P. Bouche, M. Duruflé, "High-Frequency Diffraction of a Plane Electromagnetic Wave by an Elongated Spheroid", *IEEE Transactions on Antennas and Propagation*, vol. 60, No. 11. (to appear)
- [4] I. V. Andronov, D. P. Bouche, "Forward and backward waves in high-frequency diffraction by an elongated spheroid", *PIER B*, vol. 29, p. 209–231, 2011.
- [5] V. E. Petrov, "The generalized singular Tricomi equation as a convolution equation", *Doklady Mathematics*, vol. 74, No. 3, pp. 901–905, 2006.
- [6] I. J. Thompson, A. R. Barnett, "COULCC: A continued-fraction algorithm for Coulomb functions of complex order with complex arguments", *Comput. Phys. Commun.*, vol.36, p. 363–372, 1985.