# COMPUTATION OF SINGULAR ELECTROMAGNETIC FIELDS USING A HYBRID DG-FEM METHOD

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***Abstract*—We propose a numerical method for computation of singular electromagnetic fields in a polyhedral non-convex cavity. The method is based on edge conforming elements augmented by special singular functions. The continuity of the total solution is enforced weakly via numerical fluxes. The method can be easily implemented within existing conforming finite element codes. As an application for the method we consider the computation of the smallest non-zero Maxwell eigenfrequency in a simple resonator cavity.**

## I. Introduction

The problem of finding frequencies and modes of electromagnetic waves appears in many important practical applications, such as signal processing, stability analysis etc. This problem is challenging because of the following two main reasons. The curl-operator has an infinite dimensional kernel – the gradient fields. The other difficulty occurs because of the singular behavior of the eigenmodes near re-entrant edges (or vertices) of the cavity, i.e. electromagnetic field tends to infinity. We will address only the latest issue. The natural function space for the solution of the curl-curl problem is formed by tangentially continuous edge elements, introduced by Nédélec. However, with edge finite element methods in order to reach an acceptable approximation for singular solutions, one has to use either strong mesh refinement and/or high degree polynomials [1]. A more effective approach is to employ special singular elements as introduced, e.g., by Webb [2] for tetrahedral meshes. The discretization by means of continuous elements ($H^1$) can be remedied by augmenting finite elements by special singular functions, [3] or by introducing suitable weights for a bilinear form near solution singularities, [4]. As an alternative to usual cut-off function, Lagrange multiplier techniques or solving additional boundary value problem with singular function defined over boundary, we employ the idea of DG-FEM, proposed for the 2D Laplacian problem in Classen et al [5]. Details on DG discretizations applied to Maxwell eigenvalue problems can be found in [6], [7]. In order to test the applicability of the method, we consider the Maxwell eigenvalue problems in a tensor product three-dimensional domain with perfect electric conductor (PEC) or mixed PEC and perfect magnetic conductor (PMC) boudary conditions. The first eigenpairs of these model problems coincide with the solution of corresponding 2D eigenvalue problem for the Laplace operator. The singular function can be found explicitly as a solution to the Dirichlet problem for the Laplacian in the plane sector with obtuse angle [8].

## II. Maxwell resonance frequencies

For the sake of brevity, we make the following assumptions. Let $\Omega$ be a homogeneous, isotropic, non-convex polyhedral domain with a dielectric material filing and perfectly conduction boudary $\partial\Omega$. Then the Maxwell eigenvalue problem can be written as follows. Find $\varpi \in \mathbb{R}_+$, $E \in L^2(\Omega)^3$ such that

$$\begin{aligned} \operatorname{curl}\operatorname{curl} E - \varpi^2 E &= 0 \quad \text{in } \Omega, \\ \operatorname{div} E &= 0 \quad \text{in } \Omega, \\ E \times n &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

It is known, [8], that for non-convex polyhedral domain with straight re-entrant edges the solution to this problem contains the following electrical singular functions

$$E_e^{l,1} = (\operatorname{grad}_{x,y} \Phi_e^{l,Dir}, 0) \text{ and } E_e^{l,2} = (0, 0, \Phi_e^{l,Dir}). \tag{2}$$

Here $\Phi_e^{l,Dir} = d(z_e)\, r_e^{\pi/\zeta_e} \sin\left(\frac{l\pi\theta_e}{\varphi_e}\right)$, $l \in \mathbb{N}$, is a Laplace-Dirichlet edge singularity written in cylindrical coordinates $(r_e, \theta_e, z_e)$ with axis on the edge $e$ of aperture $\zeta_e > \pi$, where $d$ is a smooth function.

Due to [9], the following decomposition of the electric (or magnetic) field holds true.

$$E = E_0 + \operatorname{grad}\varphi : \begin{cases} E_0 \in H^{1+\tau}(\Omega)^3, \ \forall \tau < \frac{\pi}{\zeta_e}, \\ \operatorname{grad}\varphi \in H^{1+\tau}(\Omega)^3, \ \forall \tau < \frac{\pi}{\zeta_e}. \end{cases}$$

Hence, the approximation will be determined by the regularity of the regular part. Depending on the choice of singular functions it can be quite low, typically $H^s$, with $s < \frac{4}{3}$ [3], [8].

## III. DG-FEM

We consider shape-regular affine meshes $\mathcal{T}_h$ that partition the domain $\Omega$ into hexahedra $K$, where the parameter $h$ denotes the mesh size. Let $\mathcal{F}_h^i$ be the set of all interior faces of



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elements in $\mathcal{T}_h$ and $\mathcal{F}_h^b$ be the set of all boundary faces. We define a finite element approximation space $\Sigma_{h,p}(\mathcal{T}_h) \in L^2(\Omega)^3$. The tangential jumps and averages across an interior face are defined by

$$[\![u]\!] = u^+ \times n^+ + u^- \times n^-, \ \{\!\{u\}\!\} = (u^+ + u^-)/2,$$

correspondingly. Then a general bilinear form for a DG method reads

$$\begin{aligned} B(E_h, v_h) &= (\operatorname{curl} E_h, \operatorname{curl} v_h) - \lambda_h(E_h, v_h) \\ &- \int_{\mathcal{F}_h^i} [\![E_h^* - E_h]\!]\{\!\{\operatorname{curl} v_h\}\!\} - \{\!\{E_h^* - E_h\}\!\}\,[\![\operatorname{curl} v_h]\!]\, ds \\ &- \int_{\mathcal{F}_h^i} [\![q_h^*]\!]\{\!\{v_h\}\!\} - \{\!\{q_h^*\}\!\}\,[\![v_h]\!]\, ds \\ &- \int_{\mathcal{F}_h^b} (n \times (E_h^* - E_h))\operatorname{curl} v_h - q_h^*(n \times v_h)\, ds \\ &\qquad \forall v_h \in \Sigma_{h,p}(\mathcal{T}_h) \end{aligned} \tag{3}$$

The derivation of a bilinear form for a DG method can be found in [10] and it follows the same ideas as the one in [11] for the Laplace operator. First of all, by introducing an auxiliary variable $q = \operatorname{curl} E \in L^2(\Omega)^3$ the Maxwell's equation (1) is rewritten as a first-order system. Then one follows the standard DG approach, i.e., both resulting equations are multiplied with arbitrary test functions and integrated by parts, the numerical fluxes are substituted in the boundary integrals and finally the second equation is again integrated by parts. Using the global lifting operators one can express the auxiliary variable and finally eliminate from the system.

Since the solution to the problem we consider is discontinuous only along an interface, the bilinear form (3) can be simplified. Let $\Gamma_e$ be a re-entrant edge. Along the edge $\Gamma_e$ we

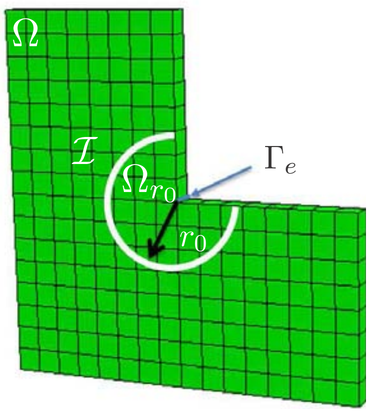


Fig. 1. Problem geometry showing a re-entrant edge domain discretized by a regular Cartesian grid. The white line shows the boundary of the cylindrical enrichment domain omega of radius r including the singular edge at the origin

define an enrichment domain $\Omega_{r_0} = \Omega \cap (B(0, r_0) \times [0, Z])$ and the interface $\mathcal{I} = \partial\Omega_{r_0} \backslash \Gamma_e$ between domains with only continuous and enriched elements, see Fig. 1. For the tangentially continuous part of the solution we choose an $H(\operatorname{curl})$-conforming finite element space $V_{h,p}(\mathcal{T}_h) \in H_0(\operatorname{curl}, \Omega)$. The approximation space $V_{h,p}(\mathcal{T}_h)$ is augmented by corresponding singular functions (2), i.e. we introduce a subspace

$$S_{r_0,\alpha} = \{E_e^{l,1}, E_e^{l,2} : l \in [1, L],\ \alpha = \pi/\zeta_e\}, \tag{4}$$

defined on $\Omega_{r_0}$. Incorporating the following numerical fluxes

$$E_h^* = (E_h|_K)_{\partial K} \quad \text{and} \quad q_h^* = (\operatorname{curl} E_h|_K)_{\partial K}$$

for (3), the proposed variational formulation reads

$$\begin{aligned} \int_\Omega \operatorname{curl} E_h \operatorname{curl} v_h\, dx - \int_{\mathcal{I}} [\![\operatorname{curl} E_h]\!]\{\!\{v_h\}\!\} - \{\!\{\operatorname{curl} E_h\}\!\}[\![v_h]\!] ds \\ = \lambda_h \int_\Omega E_h v_h\, dx \quad \forall v_h \in V_{h,p}(\mathcal{T}_h) \times S_{r_0,\alpha}. \end{aligned} \tag{5}$$

Since DG-FEM formulation (5) reduces to a standard FEM $H(\operatorname{curl})$ formulation when the enrichment domain is omitted, implementation of the proposed method based on existing FEM code is quite simple. Furthermore, the increase of number of degrees of freedom is negligible, only two new unknowns if singular functions only of the first order are used.

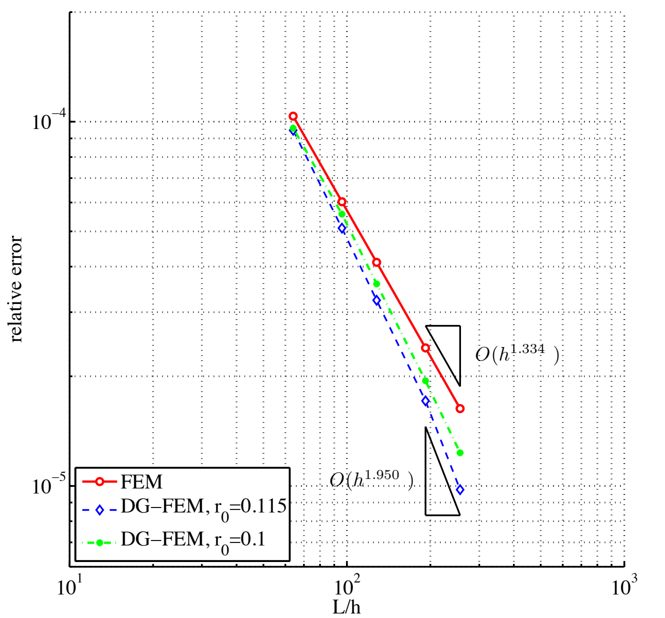


Fig. 2. Comparisons of convergence rates of the smallest non-zero Maxwell eigenfrequency ($\omega_0$ ≈148 MHz) obtained by edge conforming FEM and the DG-FEM for the thin L-shaped PEC brick

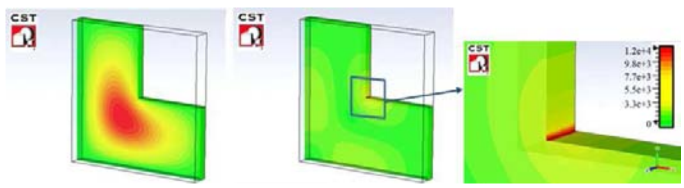

Fig. 3. Amplitude of the electric and magnetic fields corresponding to the smallest non-zero Maxwell eigenfrequency for the thin L-shaped PEC brick

## IV. Numerical examples

We present three examples which show feasibility of the proposed method and improvement of convergence order as compare to the standard edge-based FEM. Eigenvalue problems were solved numerically via Krylov-Schur method using shift and invert spectral transformation and LU factorization, as a preconditioner. Reference solutions, i.e. the first eigenvalues of the corresponding problems, are correct for at least 11 digits, see [12]. Integrals in (5) with highest singularity, i.e. $\int_\Omega r_e^{-\frac{2}{3}}\,dx$ were computed analytically. For the other integrals arising from singular functions a tensor product Gauss quadrature rule was employed.

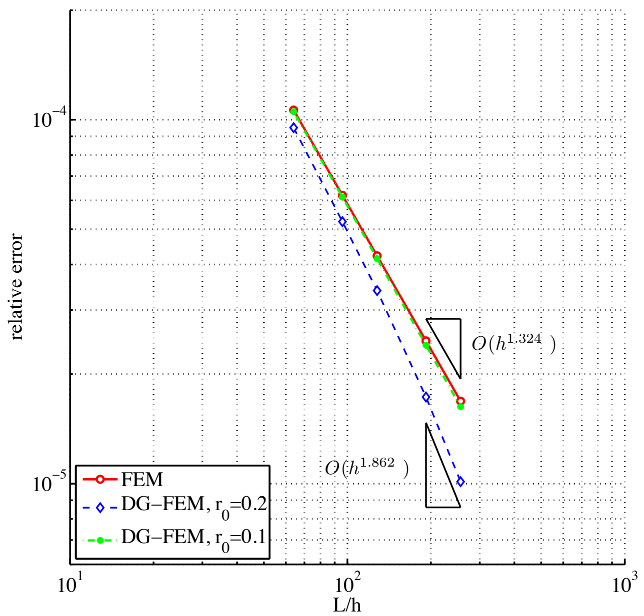


Fig. 4. Comparisons of convergence rates of the smallest non-zero Maxwell eigenfrequency ($\omega_0$ ≈58 MHz) obtained by edge conforming FEM and the DG-FEM for the thin L-shaped PMC-PEC brick

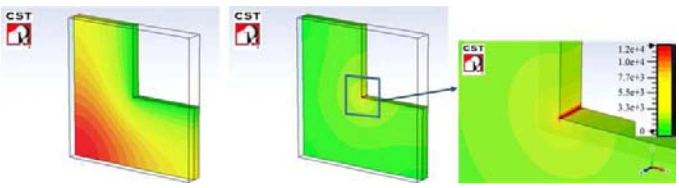

Fig. 5. Amplitude of the electric and magnetic fields corresponding to the smallest non-zero Maxwell eigenfrequency for the thin L-shaped PMC-PEC brick

Fig. 2 illustrates convergence orders obtained by FEM and DG-FEM for the first Maxwell eigenvalue in L-shaped brick with PEC boundary condition. In Fig. 3 we show amplitudes of the corresponding electric, magnetic fields and a zoomed version of the latter. In this case electric field is regular, but the magnetic field exhibits a singular behavior along the edge. Therefore the singular function of the second type $E_e^{l,2} = (0,0,\Phi_e^{l,Dir})$ is an important part of the solution.

In the second example, Fig. 4, we consider the Maxwell eigenvalue problem in the domain with mixed boundary conditions, PMC boudary condition is imposed everywhere except from two planes orthogonal to the re-entrant edge with PEC boundary. The singular functions in this case take also a form of (2) with $\Phi_e^{l,Dir}$ replaced by the Laplace-Neumann edge singularity $\Phi_e^{l,Neu}$. Since only the magnetic field has a singular behavior along the edge, as can be see from Fig. 5, the singular function of the second type $E_e^{l,2} = (0,0,\Phi_e^{l,Neu})$ gives again the contribution to the solution.

The problem geometry for the third example, Fig. 6, is reverse to the previous one, that is, PEC boudary condition is satisfied everywhere except from two planes orthogonal to the re-entrant edge with PMC boundary condition. Here the electric field is singular along the re-entrant edge, see Fig. 7, and the singular function of the first type (2) has a main impact on the solution. From the numerical experiments we observed that variational formulation (5) with the singular function of the first type demands a small modification for the enrichment subspace (4). Due to $\operatorname{curl} E_e^{l,1} = 0$, we replace $E_e^{l,1}$ by $\chi_{r_0} E_e^{l,1}$, where $\chi_{r_0}$ can be chosen as $\chi_{r_0}(r) = \sigma\, e^{-\frac{r^2}{2\sigma^2}}$,

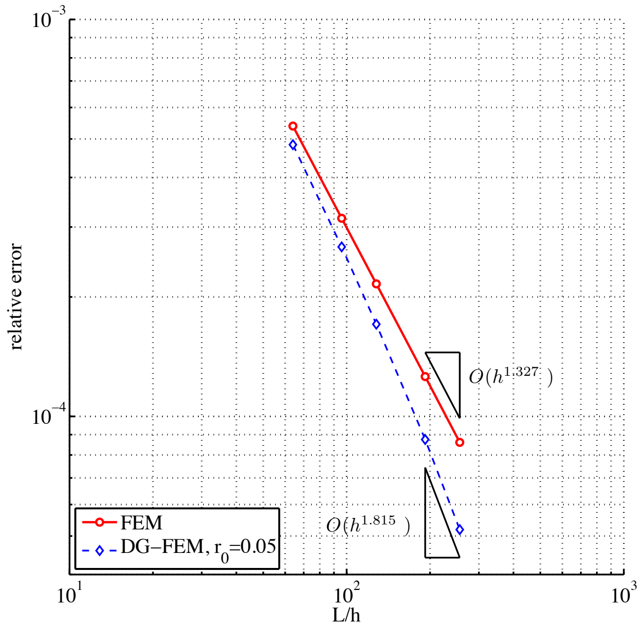


Fig. 6. Comparison of convergence rates of the smallest non-zero Maxwell eigenfrequency ($\omega_0$ ≈58 MHz) obtained by edge conforming FEM and the DG-FEM for the thin L-shaped PEC-PMC brick

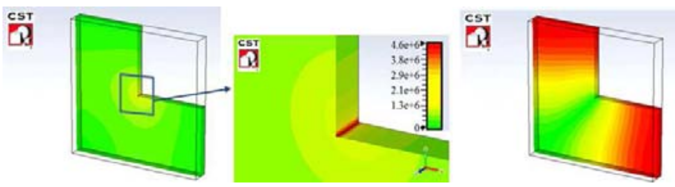

Fig. 7. Amplitude of the electric and magnetic fields corresponding to the smallest non-zero Maxwell eigenfrequency for the thin L-shaped PEC-PMC brick

$\sigma = r_0/\sqrt{2\ln 2}$. In this case, $\operatorname{curl}\chi_{r_0}E_e^{l,1} = O(r^\alpha)$. We note, that the function $\chi_{r_0}$ defined on the enrichment domain $\Omega_{r_0}$ is not a standard cut-off function since its parameters are chosen such that it represents only full width at half maximum (FWHM) part of the global cut-off function. Therefore (5) does not reduce to the variational formulation augmented by singular functions with continuity enforced via a cut-off function.

## V. Conclusion

In this study we introduced a combined DG-FEM method which incorporates modeling of both electrical and magnetic field singularities. Improvement in accuracy and convergence rate can be observed in the numerical examples. Recently Badia and Codina, [13], proposed a general combined nodal continuous-discontinuous Galerkin mixed formulation that allows arbitrary finite element spaces with functions continuous in patches of finite elements and discontinuous on the interfaces of these patches. In the future we plan to extend a DG-FEM method to a mixed variational formulation with locally singular basis functions instead of (5). Employing the mixed variational formulation will give us more freedom for choosing numerical fluxes, increase stability of the method and make the modification of the singular functions of the first type unnecessary.

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