Electromagnetic Modelling of Material Loaded Cavity Resonators with a Filling Hole for Complex Resonant Frequency Determination

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***Abstract*—The electromagnetic modelling of cavity resonators loaded with dielectric material is important with respect to corresponding material measurements. In this paper, a variational technique is employed to determine the complex resonant frequencies of the loaded cavity resonator. Inhomogeneous impedance boundary conditions are considered to take the effects of lossy walls and the filling hole of the resonator on the resonance into account. An eigenmode expansion of the magnetic field appearing in the variational formulation results in a nonlinear eigenvalue problem in terms of the complex resonant frequency. An iterative scheme is employed to solve the nonlinear eigenvalue problem. Given an initial guess, the nonlinear eigenvalue problem is approximated by a quadratic eigenvalue problem. The approximated quadratic eigenvalue problem is transformed to a generalized eigenvalue problem that is solved by using the generalized Schur decomposition. Also, the inverse problem, where the complex wavenumber is measured and the material parameter is to be determined, is discussed. In order to show the capabilities of the proposed method, the computed resonant frequencies are compared with measured data.**

I. Introduction

It is very well known that key parameters of resonators are resonant frequency and quality factor. In fact, complex eigenvalues of an electromagnetic problem are formed by these two parameters, which can also be considered as complex resonant frequencies. In the case of finite conductivity, lossy materials or radiaton losses, the self-adjoint property of the electromagnetic problem disappears. From a mathematical point of view, it results in complexification of the eigenvalues of the resonator problem.

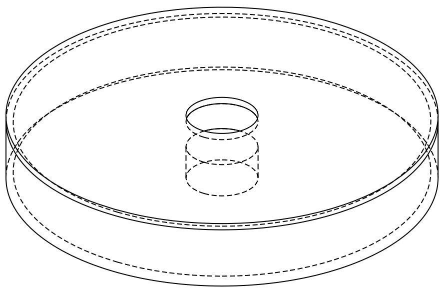
Material characterization is an important application of loaded resonators. A filling hole which may be required to place the material sample inside the cavity has an effect on the complex resonant frequencies [1], [2]. Finitely conductive walls also affect the performance of the resonator [3]. Finitely conductive walls can be modelled by an impedance boundary condition which makes the problem of finding eigenvalues of a resonator nonlinear. In [3], complex resonant frequencies of an empty resonator having lossy walls were determined by a variational technique with a reliable approximation for linearization.

In this sense, the aim of this study is to determine the complex resonant frequencies of a cavity resonator having lossy walls, a filling hole, as well as dielectric loading and estimate material parameters when the measured complex resonant frequency is given. The problem of finding the complex eigenvalues of the loaded resonator is formulated as variational problem [3]. The finitely conductive walls are modelled by impedance boundaries. Under appropriate assumptions, the filling hole is also modelled by an artificial impedance wall. Its impedance is chosen as the wave impedance of an appropriate evanescent mode of a waveguide having the same cross-section as the filling hole. Then, the impedance of the lossy walls and of the artificial impedance of the filling hole makes the surface impedance a piecewise constant function. The magnetic field is expanded in terms of the natural modes of the cavity in order to reduce the variational expression into matrix form. The resulting eigenvalue problem, where the material parameters are given and the complex resonant frequency is to be obtained, is nonlinear. This problem is called the direct problem. The problem, where the material parameter is to be obtained by using the measured complex resonant frequency is called the inverse problem and is a generalized eigenvalue problem. In order to solve the direct problem, it is approximated by a quadratic form. Then, it is transformed into a generalized eigenvalue problem by substituting a new variable [4]. The generalized eigenvalue problem is solved by using the generalized Schur decomposition [5]. This scheme is employed iteratively until a predefined stopping criteria is satisfied. In the inverse problem, the material parameters are estimated by applying the generalized Schur decomposition. In order to demonstrate the capabilities of the proposed method, a comparison of the complex resonant frequencies with some experimental data is presented. The complex relative permittivities of teflon and compressed and bound silicon carbide powder are presented.

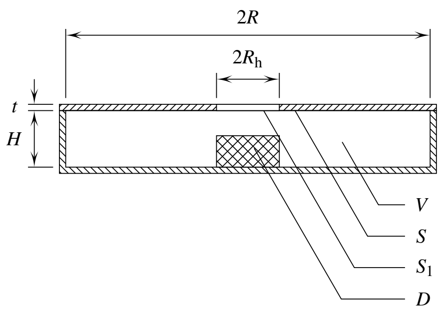
II. Formulation

A. The direct problem

Consider the geometry shown in Fig. 1, where a circular cavity V with a boundary surface S is loaded with a non-magnetic ($\mu = \mu_0$) object D. It has complex permittivity $\varepsilon_{\mathrm{d}}\varepsilon_0$, where ε_0 is the permittivity of vacuum. The background



(a) Perspective view



(b) Cross-sectional view

Figure 1. Cylindrical cavity with dielectric probe.

medium $V \setminus \bar{D}$ is assumed to be vacuum ($\varepsilon = \varepsilon_0$). Hence, the relative permittivity is

$$\varepsilon_\mathrm{r}(\boldsymbol{r}) = \begin{cases} \varepsilon_\mathrm{d} & \text{for} \quad \boldsymbol{r} \in D \\ 1 & \text{for} \quad \boldsymbol{r} \in V \setminus D\,, \end{cases} \tag{1}$$

where $\boldsymbol{r}$ is the position of any point inside the cavity. The boundary surface $S \setminus S_1$ represents the conductive walls. The circular filling hole that is used to plunge the dielectric material sample into the cavity is denoted by surface S_1. The cavity walls shall be thick and the hole diameter shall be below cutoff such that the field inside the hole decays rapidly. Under these assumptions, one can define the boundary of the cavity as a piecewise constant impedance

$$Z(\boldsymbol{r}) = \begin{cases} Z_\mathrm{w} & \text{for} \quad r \in S_1 \\ Z_\mathrm{s} & \text{for} \quad r \in S \setminus S_1\,. \end{cases} \tag{2}$$

Here, Z_s and Z_w are the surface impedance of the finitely conductivity walls and the wave impedance of an appropriate evanescent mode of a waveguide having the same cross-section as the filling hole, respectively. Then, the magnetic field $\boldsymbol{H}$ satisfies the vector wave equation

$$\nabla \times \frac{1}{\varepsilon_\mathrm{r}(\boldsymbol{r})} \nabla \times \boldsymbol{H} - k^2 \boldsymbol{H} = 0 \tag{3}$$

and is subject to the boundary condition

$$\boldsymbol{n} \times \nabla \times \boldsymbol{H} = -\mathrm{j} Y_0 k Z(\boldsymbol{r}) \boldsymbol{H}^\mathrm{t}\,, \tag{4}$$

where $\boldsymbol{H}^\mathrm{t}$, Y_0 $\boldsymbol{n}$ and k denote the tangential component of the magnetic field, the intrinsic admittance of vacuum, the unit inward normal to the surface S, and the complex wavenumber to be determined, respectively. The problem of finding the complex eigenvalues k of (3) subject to the approximate boundary condition (4) can be converted into a variational problem. One can take the inner product between the wave equation (3) and $\boldsymbol{H}$, use Green's first identity, and impose the boundary condition (4) in order to obtain the functional

$$\boldsymbol{I} = \int_V [(\nabla \times \boldsymbol{H}) \cdot \frac{1}{\varepsilon_\mathrm{r}} (\nabla \times \boldsymbol{H}) - k^2 \boldsymbol{H} \cdot \boldsymbol{H}]\, \mathrm{d}V + \mathrm{j} k Y_0 \oint_S Z \boldsymbol{H}^\mathrm{t} \cdot \boldsymbol{H}^\mathrm{t}\, \mathrm{d}S, \tag{5}$$

which is a variational expression for the complex wavenumber k. Since the undamped modes $\boldsymbol{H}_n$ of the empty cavity form a complete set, the magnetic field $\boldsymbol{H}$ appearing in (5) is expanded into the undamped modes $\boldsymbol{H}_n$

$$\boldsymbol{H} = \sum_{n=1}^{N} \alpha_n \boldsymbol{H}_n\,. \tag{6}$$

Here, α_n are the unknown coefficients of the undamped modes. One can reduce the problem to matrix form by inserting (6) in (5) and equalling the first variation of the functional with respect to the unknown coefficients α_n to zero

$$(A_{mn} k^2 + B_{mn}(k) k + C_{mn} - \nu(\varepsilon_\mathrm{d}) E_{mn})(\alpha_n) = 0, \quad n, m = 1, 2, \ldots, N\,, \tag{7}$$

where

$$A_{mn} = \int_V \boldsymbol{H}_n \cdot \boldsymbol{H}_m\, \mathrm{d}V \tag{8}$$

$$B_{mn} = \mathrm{j} Y_0 \oint_S Z \boldsymbol{H}_n^\mathrm{t} \cdot \boldsymbol{H}_m^\mathrm{t}\, \mathrm{d}S \tag{9}$$

$$C_{mn} = -k_n k_m \delta_{nm} \tag{10}$$

$$E_{mn} = \int_D (\nabla \times \boldsymbol{H}_n) \cdot (\nabla \times \boldsymbol{H}_m) \mathrm{d}V \tag{11}$$

$$\nu(\varepsilon_\mathrm{d}) = \left(\frac{1}{\varepsilon_\mathrm{d}} - 1\right)\,. \tag{12}$$

Equation (7) is a nonlinear eigenvalue problem in terms of the complex wavenumber k. Here, $\boldsymbol{B} = [B_{mn}]$ and $\boldsymbol{E} = [E_{mn}]$ are symmetric and can be evaluated numerically or analytically. $\boldsymbol{A} = [A_{mn}]$ is the identity matrix. k_n and δ_{mn} denote the real wavenumber of the lossless empty cavity belonging to the modal field $\boldsymbol{H}_n$ and the Kronecker delta, respectively. Given an initial guess k_0, the nonlinear eigenvalue problem is approximated by a quadratic one

$$Q(k_1) = \big(A k_1^2 + B(k_0) k_1 + C\big) \alpha = 0. \tag{13}$$

The N dimensional quadratic eigenvalue problem can be converted into a $2N$ generalized eigenvalue problem by substituting $u = k_1 \alpha$ [4]

$$\begin{bmatrix} 0 & I \\ -C + \nu E & -B(k_0) \end{bmatrix} \begin{bmatrix} \alpha \\ u \end{bmatrix} - k_1 \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \alpha \\ u \end{bmatrix} = 0\,, \tag{14}$$

which is solved by the generalized Schur decomposition [5]. Once k_1 is obtained, $\boldsymbol{B}$ is updated as $\boldsymbol{B}(k_1)$ and the quadratic problem is solved again. This iteration scheme is continued until a desired level of accuracy is obtained.

B. The inverse problem

With the measured resonant frequency and the quality factor of the loaded resonator cavity, the complex relative permittivity of the material placed in the resonator can be estimated To this aim, one can rearrange equation (7) in order to solve the inverse problem associated with the loaded resonator cavity [6]

$$(A_{mn}k^2 + B_{mn}(k)k + C_{mn})(\alpha_n) = \nu(\varepsilon_\mathrm{d})E_{mn}(\alpha_n), \quad n, m = 1, 2, \ldots, N\,, \tag{15}$$

which is a generalized eigenvalue problem for the unknown $\nu(\varepsilon_\mathrm{d})$. The sample holder D_h as an additional material may be required to keep the material to be measured in a desired position for some cases. In this case, a new matrix $\boldsymbol{F}$ has to been inserted in (15) resulting in

$$(A_{mn}k^2 + B_{mn}(k)k + C_{mn} - F)(\alpha_n) = \nu(\varepsilon_\mathrm{d})E_{mn}(\alpha_n), \quad n, m = 1, 2, \ldots, N\,, \tag{16}$$

where

$$F_{mn} = \nu(\varepsilon_\mathrm{s}) \int_{D_\mathrm{h}} (\nabla \times \boldsymbol{H}_m) \cdot (\nabla \times \boldsymbol{H}_n)\,\mathrm{d}V. \tag{17}$$

Here, ε_s is the known relative permittivity of the sample holder D_h. The generalized Schur decomposition can be applied to (15) or (16) in order to estimate the complex relative permittivity ε_d of the sample under test.

III. Results

The dimensions of the cavity are as follows: radius $R = 45\,\mathrm{mm}$, height $H = 13.7\,\mathrm{mm}$, radius of the filling hole $R_\mathrm{h} = 6\,\mathrm{mm}$, and thickness of the walls $t = 6\,\mathrm{mm}$. The conductivity of the lossy walls was $\sigma = 2.47 \times 10^7\,\mathrm{S/m}$ (brass). The measured complex resonant frequency was formed by the measured resonant frequency f_r and the unloaded quality factor Q_u,

$$f_\mathrm{m} \approx f_\mathrm{r}\left(1 + \mathrm{j}\frac{1}{2Q_\mathrm{u}}\right). \tag{18}$$

Firstly, the direct algorithm was tested. To this aim, a cylindrical teflon plug with radius $6\,\mathrm{mm}$ was used to load the cavity resonator. The magnetic field was expanded in TM_{0mn} modes. The number of basis functions was $N = 500$. TM_{010} and TM_{020} modes were excited by coupling structures. Therefore, the TM_{01} mode was the most influential one inside the circular filling hole. Z_w was chosen as the wave impedance of the TM_{01} mode of the artificial circular waveguide. The field associated to the TM_{01} mode inside the hole falls to $1/\mathrm{e}$ in $2.52\,\mathrm{mm}$ for the operating frequency $f = 2.55\,\mathrm{GHz}$. Fig. 2 and Fig. 3 demonstrate the amplitude of the electric field distributions at $\phi = 0$ by neglecting the effect of the filling and considering it, respectively. In Table I, a comparison of the measured parameters and the computed parameters by the proposed method is given for two different cases: 1) neglecting the effect of the hole and 2) considering it. From Table I, Fig. 2 and Fig. 3, it can be seen that the filling hole has an effect on the field distribution and, therefore, also on the complex resonant frequency. One can also conclude that the method is quite capable of determining the complex resonant frequencies accurately.

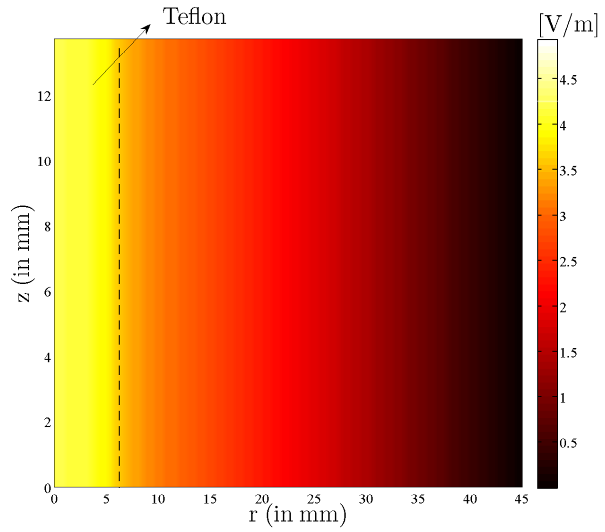


Figure 2. Electric field distribution of the cavity loaded with teflon for the case the effect of the filling hole is not considered.

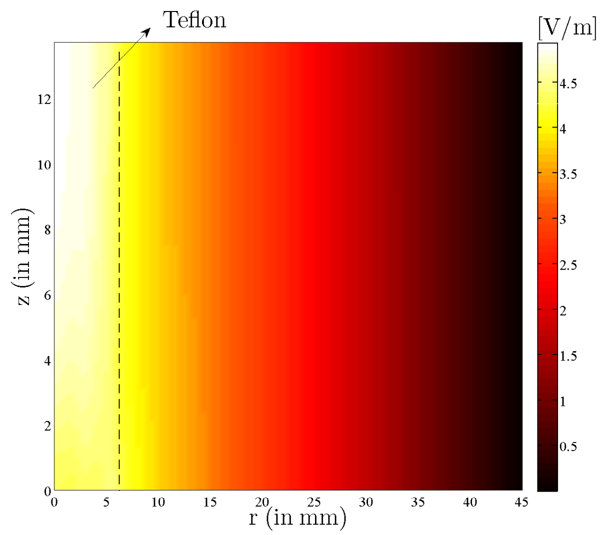


Figure 3. Electric field distribution of the cavity loaded with teflon for the case the effect of the filling hole is considered.

In order to test the inversion algorithm, the measured resonant frequencies of the cavity resonator loaded with teflon were used to estimate the complex relative permittivity of teflon by taking the effect of the hole and neglecting it. Table II shows that the method is capable of determining material parameters accurately when the effect of the filling hole is considered. Finally, the temperature dependent measurement of compressed and bound silicon carbide powder is given in Table III. The measurement setup is shown in Fig. 4 where two additional materials with known parameters were placed

Table I
THE MEASURED AND COMPUTED COMPLEX RESONANT FREQUENCIES NEGLECTING AND CONSIDERING THE EFFECT OF THE FILLING HOLE

mode	loading	measurement		proposed method neglecting the filling hole			proposed method considering the filling hole		
		$\mathrm{Re}\{f_\mathrm{m}\}$	$\mathrm{Im}\{f_\mathrm{m}\}$	$\mathrm{Re}\{f_\mathrm{c}\}$	$\mathrm{Im}\{f_\mathrm{c}\}$	relative error	$\mathrm{Re}\{f_\mathrm{c}\}$	$\mathrm{Im}\{f_\mathrm{c}\}$	relative error
TM_{010}	empty	2.565e9 Hz	3.644e5 Hz	2.551e9 Hz	3.493e5 Hz	0.55 %	2.558e9 Hz	3.502e5 Hz	0.28 %
	teflon	2.497e9 Hz	3.816e5 Hz	2.462e9 Hz	3.536e5 Hz	1.40 %	2.483e9 Hz	3.531e5 Hz	0.56 %
TM_{020}	empty	5.916e9 Hz	4.820e5 Hz	5.856e9 Hz	5.293e5 Hz	1.06 %	5.886e9 Hz	5.261e5 Hz	0.56 %
	teflon	5.580e9 Hz	3.678e5 Hz	5.439e9 Hz	5.686e5 Hz	2.53 %	5.533e9 Hz	5.438e5 Hz	0.85 %

Table II
THE ESTIMATED RELATIVE PERMITTIVIES OF TEFLON MATERIAL BY NEGLECTING AND CONSIDERING THE EFFECT OF THE FILLING HOLE

frequency	material	reference		proposed method neglecting the filling hole			proposed method considering the filling hole		
		$\mathrm{Re}\{\varepsilon_\mathrm{r}\}$	$-\mathrm{Im}\{\varepsilon_\mathrm{r}\}$	$\mathrm{Re}\{\varepsilon_\mathrm{r}\}$	$-\mathrm{Im}\{\varepsilon_\mathrm{r}\}$	relative error	$\mathrm{Re}\{\varepsilon_\mathrm{r}\}$	$-\mathrm{Im}\{\varepsilon_\mathrm{r}\}$	relative error
2.55 GHz	teflon	2.100	0.00042	2.011	0.00011	4.22 %	2.021	0.00016	3.74 %
5.86 GHz	teflon	2.100	0.00042	1.751	0.00059	16.6 %	2.078	0.00091	1.05 %

Table III
THE ESTIMATED RELATIVE PERMITTIVITY OF COMPRESSED AND BOUND SILICON CARBIDE POWDER

Averaged Temperature °C	20	115	209	285	354	461	580
Uncertainty in Temperature °C	0	15	24	35	56	72	101
$\mathrm{Re}\{\varepsilon_\mathrm{r}\}$	8.00	9.91	9.04	10.14	11.65	12.38	11.36
$-\mathrm{Im}\{\varepsilon_\mathrm{r}\}$	2.10	0.94	0.49	0.55	0.61	0.91	1.01

inside the cavity for support and thermal insulation. Since the compressed and bound silicon carbide powder is a lossy material, the support material is used to reduce the height of the sample inside the cavity in order to keep the quality factor relatively high. The sample was heated to a certain temperature and then removed from the oven. Then, it was placed inside the cavity through the filling hole and measured immediately.

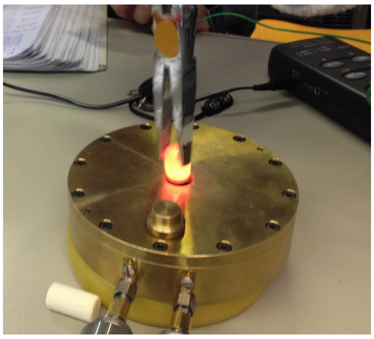


Figure 4. Temperature-dependent measurement setup.

IV. CONCLUSION

The direct and inverse problems associated with a cavity resonator loaded with isotropic, dielectric material were addressed. The effects of the filling hole and the conductive walls were examined by modelling them as inhomogeneous impedance boundary conditions. The problem was formulated by using the variational expression inherited from the vector wave equation for the magnetic field. The magnetic field was expanded into eigenmodes of the empty cavity in order to reduce the problem into matrix form. Given an initial guess, the resulting nonlinear eigenvalue problem was approximated by a quadratic eigenvalue problem. This problem in turn was solved iteratively to determine the complex resonant frequencies of the cavity. In the case the complex wavenumber was measured and the complex relative permittivity of the loaded material was to be found, the inverse problem was considered. It was solved by using the generalized Schur decomposition. The computed parameters were compared with the measured parameters. Also, the inverse algorithm was tested against the measured data in order to estimate the complex relative permittivity of teflon. Promising results were reported.

REFERENCES

[1] E.H. Bussey, and A.J. Estin, “Errors in Dielectric Measurements Due to a Sample Insertion Hole in a Cavity”, *IRE Transactions on Microwave Theory and Techniques*, vol. 8, no. 6, pp. 650–653, November 1960.

[2] S. Gauthier, L. Marchildon, and C. Akyel, “Shift of the Complex Resonance Frequency of a Dielectric-Loaded Cavity Produced by Small Sample Insertion Holes”, *IEEE Transactions on Microwave Theory and Techniques*, vol. 37, no. 4, pp. 801–804, April 1989.

[3] R. E. Collin, *Field Theory of Guided Waves*, 2nd edition, New York, Wiley-Interscience, 1991.

[4] F. Tisseur, and K. Meerbergen, “The Quadratic Eigenvalue Problem”, *SIAM Review*, vol. 43, no. 2, pp. 235–286, 2001.

[5] R. A. Horn and R. C. Johnson, *Matrix Analysis*, 1st edition, Cambridge, Cambridge University Press, 1985.

[6] E. Kılıç, U. Siart, O. Wiedenmann, U. Faz, R. Ramakrishnan, P. Saal and T. F. Eibert, ”Cavity Resonator Measurement of Dielectric Materials Accounting for Wall Losses and a Filling Hole,” *IEEE Transactions on Instrumentation and Measurement,* accepted, 2012.