Singularities or Emergent Losses in Radially Uniaxial Spheres

Henrik Wallén*, Henrik Kettunen, Ari Sihvola

Aalto University School of Electrical Engineering
Espoo, Finland
*henrik.wallen@aalto.fi

***Abstract*—To find the electrostatic solution of a radially uniaxially anisotropic sphere in a constant external field is fairly straightforward. However, the resulting quasistatic polarizability exhibits many interesting properties including singularities and dubious emergent losses. By removing the origin and adding small losses, it turns out that the solution is valid in a certain limiting sense. Without puncturing the sphere, the potential is singular at the origin if the permittivity is indefinite, and for certain ranges of parameter values infinitesimally small losses give rise to significant effective losses.**

I. Introduction

The static solution involving a radially uniaxial (RU) sphere has been considered in several publications [1]–[5] and also the Mie-scattering from an RU sphere has been solved [6]. In this presentation we continue the work by carefully analyzing the validity of the electrostatic solution when the permittivity components are allowed to be negative. Especially in the indefinite case, when the two permittivity components have opposite signs, the solution contains very interesting properties, which at first sight appear totally nonsensical. However, the simple but dubious solution appears to be valid as the result of a limiting process involving infinitesimally small losses and a vanishing perfectly conducting core.

II. Quasistatic Potential and Polarizability

A. RU Sphere

Consider a sphere with radius a and the radially uniaxial permittivity

$$\overline{\overline{\varepsilon}} = \varepsilon_0 \left[\varepsilon_{\mathrm{rad}}\, \mathbf{u}_r \mathbf{u}_r + \varepsilon_{\tan} \left(\mathbf{u}_\theta \mathbf{u}_\theta + \mathbf{u}_\varphi \mathbf{u}_\varphi\right)\right] \tag{1}$$

centered at the origin of a spherical coordinate-system (r, θ, φ). The external field is oriented along the z-axis, i.e., the external potential is

$$\phi_0(r, \theta) = -U_0 \frac{r}{a} \cos\theta. \tag{2}$$

The solution must be independent of φ, since neither the geometry nor the excitation depends on φ. The θ-dependence, both inside and outside the sphere, is of the form $P_n(\cos\theta)$, where P_n is the Legendre polynomial and n is a non-negative integer. However, due to the orthogonality of the Legendre polynomials and the form of the excitation, only $P_1(\cos\theta) = \cos\theta$ is needed in the solution.

The potential outside the sphere, $r > a$, must approach ϕ_0 as $r \to \infty$, and so the solution must be of the form

$$\phi_{\mathrm{out}}(r, \theta) = U_0 \frac{\alpha}{3} \left(\frac{r}{a}\right)^{-2} \cos\theta - U_0 \frac{r}{a} \cos\theta, \tag{3}$$

where the strength of the dipolar term is expressed using the *normalized polarizability* α.

The potential in a radially uniaxial medium satisfies the (generalized) Laplace equation

$$\nabla \cdot \left(\overline{\overline{\varepsilon}} \cdot \nabla\phi\right) = 0, \tag{4}$$

whose general solution can easily be found using separation of variables in spherical coordinates. The angular part contains the usual spherical harmonics, which reduce into Legendre polynomials $P_n(\cos\theta)$ in the φ-independent case, but the radial part is more peculiar. In the present case, where only the solutions with $\cos\theta$ angular dependence is excited, we can express the general potential solution in the form

$$\phi(r, \theta) = \left[A \left(\frac{r}{a}\right)^{\nu} + B \left(\frac{r}{a}\right)^{-\nu-1}\right] \cos\theta, \tag{5}$$

where

$$\nu = \frac{1}{2}\left(-1 + \sqrt{1 + 8\varepsilon_{\tan}/\varepsilon_{\mathrm{rad}}}\right). \tag{6}$$

The potential inside the sphere, $r < a$, should be finite at the origin, and so we choose

$$\phi_{\mathrm{in}}(r, \theta) = A \left(\frac{r}{a}\right)^{\nu} \cos\theta. \tag{7}$$

Enforcing the interface conditions

$$\phi_{\mathrm{in}} = \phi_{\mathrm{out}}, \qquad \varepsilon_{\mathrm{rad}} \frac{\partial\phi_{\mathrm{in}}}{\partial r} = \frac{\partial\phi_{\mathrm{out}}}{\partial r}, \qquad \text{when } r = a, \tag{8}$$

we get the normalized polarizability in the familiar form

$$\alpha = 3\frac{\varepsilon_{\mathrm{eff}} - 1}{\varepsilon_{\mathrm{eff}} + 2}, \tag{9}$$

where the effective permittivity is

$$\varepsilon_{\mathrm{eff}} = \varepsilon_{\mathrm{rad}} \nu = \frac{\varepsilon_{\mathrm{rad}}}{2}\left(-1 + \sqrt{1 + 8\varepsilon_{\tan}/\varepsilon_{\mathrm{rad}}}\right), \tag{10}$$

and the amplitude of the internal potential is

$$A = \frac{-3U_0}{\varepsilon_{\mathrm{eff}} + 2}. \tag{11}$$

Since the normalized polarizability (9) is exactly the same as for a homogeneous sphere with effective permittivity $\varepsilon_{\mathrm{eff}}$, this

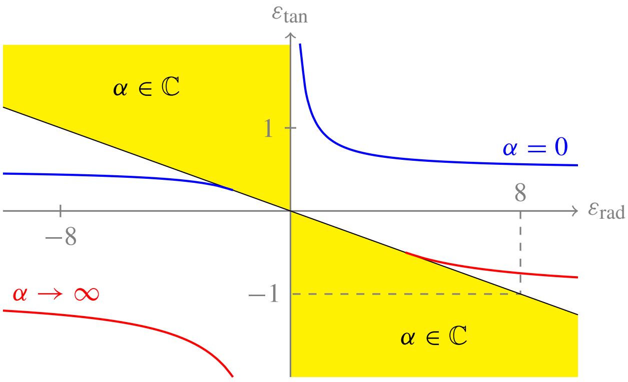

Fig. 1. The polarizability of an ideal RU sphere is well defined in the first and third quadrants of the $\varepsilon_{\mathrm{rad}}\,\varepsilon_{\tan}$-plane. The potential at the origin is singular if $\varepsilon_{\mathrm{rad}}$ and $\varepsilon_{\tan}$ have opposite signs, and in the yellow regions the potential and polarizability are complex. Also shown are the lines where the polarizability vanishes (blue lines in the upper half-plane) and where the polarizability is singular (red lines in the lower half-plane).

solution can also be interpreted as an *internal homogenization* [7].

The validity of this solution depends on the anisotropy ratio $\varepsilon_{\tan}/\varepsilon_{\mathrm{rad}}$ of the RU sphere. If the ratio is real, we can separate three distinct cases based on the exponent ν given by (6):

1) If $\varepsilon_{\tan}/\varepsilon_{\mathrm{rad}} > 0$, the potential is finite at the origin, and there does not seem to be any problem with the solution.
2) If $-1/8 < \varepsilon_{\tan}/\varepsilon_{\mathrm{rad}} < 0$, the square root in (6) is real, but the exponent $\nu < 0$, and so the potential (7) is singular at the origin. The polarizability (9) does still give a real and possibly reasonable value.
3) If $\varepsilon_{\tan}/\varepsilon_{\mathrm{rad}} < -1/8$, the exponent ν is complex with a negative real part, and so the potential is singular and oscillatory at the origin. The polarizability is finite but complex.

These regions are visualized in Fig. 1, which also shows the locations where the polarizability vanishes

$$\varepsilon_{\tan} = \frac{\varepsilon_{\mathrm{rad}}+1}{2\varepsilon_{\mathrm{rad}}}, \qquad \varepsilon_{\mathrm{rad}} < -2,\ \ \varepsilon_{\mathrm{rad}} > 0, \tag{12}$$

and where the polarizability is singular

$$\varepsilon_{\tan} = \frac{2-\varepsilon_{\mathrm{rad}}}{\varepsilon_{\mathrm{rad}}}, \qquad \varepsilon_{\mathrm{rad}} < 0,\ \ \varepsilon_{\mathrm{rad}} > 4. \tag{13}$$

Notice that the RU sphere can be invisible while the potential is singular at the origin at the same time, according to (12) with $\varepsilon_{\mathrm{rad}} < -2$.

B. Punctured RU Sphere

Due to symmetry considerations, the potential should be zero at the origin. Thus, it seems reasonable to remove the singularity at the origin by inserting a grounded perfectly conducting (PEC) sphere with radius $b < a$ at the origin and thereafter letting $b/a \to 0$.

The potential outside the RU sphere, $r > a$, is still of the form (3) and the potential inside, $b < r < a$, is of the form (5). The normalized polarizability α and the coefficients A and B can be solved by enforcing the boundary condition $\phi_{\mathrm{in}} = 0$ at $r = b$ and the interface conditions (8).

After some straightforward but somewhat lengthy calculations, we can express the polarizability in the familiar form

$$\alpha = 3\frac{\varepsilon_{\mathrm{eff}}-1}{\varepsilon_{\mathrm{eff}}+2}, \tag{14}$$

where the effective permittivity is

$$\varepsilon_{\mathrm{eff}} = \frac{\varepsilon_{\mathrm{rad}}}{2}\left(-1+\sqrt{\cdot}\,\frac{1+\left(\frac{b}{a}\right)^{\sqrt{\cdot}}}{1-\left(\frac{b}{a}\right)^{\sqrt{\cdot}}}\right), \tag{15}$$

where

$$\sqrt{\cdot} = \sqrt{1+8\varepsilon_{\tan}/\varepsilon_{\mathrm{rad}}} = 2\nu+1. \tag{16}$$

The coefficients A and B for the potential (5) inside the punctured RU sphere are

$$A = \frac{-3U_0}{(\varepsilon_{\mathrm{eff}}+2)\left(1-\left(\frac{b}{a}\right)^{\sqrt{\cdot}}\right)}, \qquad B = -\left(\frac{b}{a}\right)^{\sqrt{\cdot}} A. \tag{17}$$

This solution for the punctured RU sphere is real and finite for any $b > 0$ and $\varepsilon_{\mathrm{rad}}, \varepsilon_{\tan} \in \mathbb{R}$. The imaginary parts cancel out exactly, although it is not immediately obvious from the above formulas.

If the square root (16) is real, i.e., $\varepsilon_{\tan}/\varepsilon_{\mathrm{rad}} > -1/8$, the limit

$$\lim_{b\to 0}\left(\frac{b}{a}\right)^{\sqrt{\cdot}} = 0 \tag{18}$$

is well defined and we get the RU solution when the PEC core vanishes. Thus, the RU solution can be considered valid also in the region $-1/8 < \varepsilon_{\mathrm{rad}}/\varepsilon_{\tan} < 0$.

If $\varepsilon_{\mathrm{rad}}/\varepsilon_{\tan} < -1/8$, the square root (16) is purely imaginary and the limit does not exist since

$$\left(\frac{b}{a}\right)^{\pm j\beta} = \cos\left(\beta \ln\frac{b}{a}\right) \pm j\sin\left(\beta\ln\frac{b}{a}\right), \tag{19}$$

which oscillates without reaching a limit when $b/a \to 0$.

Fig. 2 shows an example of the striking differences in the polarizability of an RU sphere with and without a very small PEC-core. The results should agree in the limit $b/a \to 0$ if the solution is unique, and this is clearly true for $\varepsilon_{\mathrm{rad}}/\varepsilon_{\tan} > 0$ and $\varepsilon_{\mathrm{rad}}/\varepsilon_{\tan} < -8$. For permittivity components falling into the complex (yellow) region in Fig. 1, we either get singularities or a complex (lossy) polarizability from real (lossless) material. Neither solution seems reasonable, but the ambiguity is very similar to the branch cut or unlimited number of singularities that arise in geometries containing sharp corners and certain ranges of negative permittivity [8]–[11].

C. Losses

Adding losses makes the limit $b/a \to 0$ well defined. Using the time convention $e^{+j\omega t}$, small losses as a negative imaginary part of either $\varepsilon_{\mathrm{rad}}$ and $\varepsilon_{\tan}$ (or both) ensure that the real part of the square root (16) is nonzero, which is sufficient to make the limit (18) vanish. Therefore, the punctured RU result approaches the RU one as long as there are some

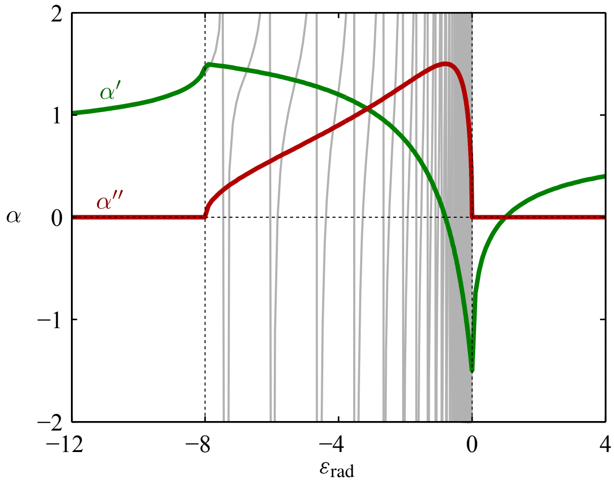

Fig. 2. Polarizability $\alpha' - j\alpha''$ of an ideal RU sphere with $\varepsilon_{\mathrm{tan}} = 1$ as a function of $\varepsilon_{\mathrm{rad}}$ compared with the polarizability of a punctured RU sphere with $b/a = 10^{-10}$ (gray line).

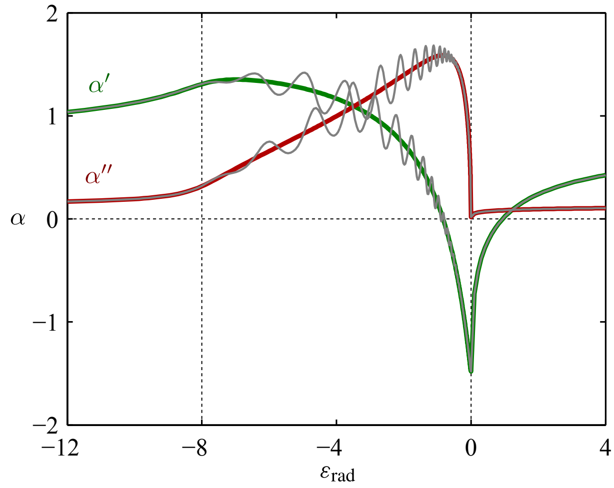

Fig. 3. The polarizability, as in Fig. 2, with losses added to the tangential permittivity $\varepsilon_{\mathrm{tan}} = 1 - j/10$. The polarizability of the punctured RU sphere with $b/a = 10^{-10}$ is again plotted using thin gray lines. Smaller b/a or larger losses makes the agreement between the punctured and non-punctured cases better.

material losses involved. This argument is valid for arbitrarily small losses, and thus it seems that the strange complex RU polarizability for $\varepsilon_{\mathrm{tan}}/\varepsilon_{\mathrm{rad}} < -1/8$ is valid after all. However, the result for real permittivity components must be understood as an approximation where the losses are infinitesimally small.

Fig. 3 shows the polarizability of a lossy RU sphere compared with the same sphere with a very small PEC core. Although the results coincide exactly in the limit $b/a \to 0$, the PEC core must be ridiculously small to numerically get good agreement, unless the losses are large.

III. Conclusions

The normalized polarizability (9) and effective permittivity (10) of a radially uniaxial (RU) sphere are easily found, but the results appear to be dubious at first sight when the permittivity (1) is indefinite. The results are especially troublesome in the region $\varepsilon_{\mathrm{tan}}/\varepsilon_{\mathrm{rad}} < -1/8$, where the potential is singular and the polarizability is complex.

To get a unique and well behaved solution for all possible parameters $\varepsilon_{\mathrm{tan}}$ and $\varepsilon_{\mathrm{rad}}$, we assume that there are some losses in the material and remove the singularity at the origin by inserting a grounded PEC core. The main result is that the potential solution and polarizability tend to the RU solution in the limit when the core vanishes, for arbitrarily small losses.

However, the complex polarizability for real permittivity components should be interpreted carefully. Absorption enhancement, instead of emergent losses, is a physically more reasonable explanation, when very small losses in the material can give large effective losses in the RU sphere.

References

[1] K. Schulgasser, “Sphere assemblage model for polycrystals and symmetric materials,” *J. Appl. Phys.*, vol. 54, no. 3, pp. 1380–1382, Mar. 1983.

[2] J. C. de Munck, “The potential distribution in a layered anisotropic spheroidal volume conductor,” *J. Appl. Phys.*, vol. 64, no. 2, pp. 464–470, Jul. 1988.

[3] J. Helsing and A. Helte, “Effective conductivity of aggregates of anisotropic grains,” *J. Appl. Phys.*, vol. 69, no. 6, pp. 3583–3588, Mar. 1991.

[4] J. C.-E. Sten, “DC fields and analytical image solutions for a radially anisotropic spherical conductor,” *IEEE Trans. Dielectr. Electr. Insul.*, vol. 2, no. 3, pp. 360–367, Jun. 1995.

[5] T. Rimpiläinen, H. Wallén, H. Kettunen, and A. Sihvola, “Electrical response of systropic sphere,” *IEEE Trans. Antennas Propag.*, vol. 60, no. 11, pp. 5348–5355, Nov. 2012.

[6] C. Qiu, L. Gao, J. D. Joannopoulos, and M. Soljačić, “Light scattering from anisotropic particles: propagation, localization, and nonlinearity,” *Laser & Photon. Rev.*, vol. 4, no. 2, pp. 268–282, 2010.

[7] U. K. Chettiar and N. Engheta, “Internal homogenization: Effective permittivity of a coated sphere,” *Opt. Express*, vol. 20, no. 21, pp. 22 976–22 986, Oct. 2012.

[8] H. Kettunen, H. Wallén, and A. Sihvola, “Electrostatic resonances of a negative-permittivity hemisphere,” *J. Appl. Phys.*, vol. 9, no. 103, pp. 094 112:1–8, May 2008.

[9] H. Wallén, H. Kettunen, and A. Sihvola, “Surface modes of negative-parameter interfaces and the importance of rounding sharp corners,” *Metamaterials*, vol. 2, no. 2–3, pp. 113–121, 2008.

[10] M. Pitkonen, “A closed-form solution for the polarizability of a dielectric double half-cylinder,” *J. Electromagn. Waves Appl.*, vol. 24, no. 8–9, pp. 1267–1277, 2010.

[11] J. Helsing, R. C. McPhedran, and G. W. Milton, “Spectral super-resolution in metamaterial composites,” *New J. Phys.*, vol. 13, no. 11, pp. 115 005:1–17, 2011.