A Simple Expression for Bit Error Probability of Convolutional Codes under Class-A Interference

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***Abstract*— In order to establish emission limits to protect radio systems, the effect of the non-Gaussian properties of interference on the performance of coded radio transmission systems should be taken into account. In this paper, a simple closed-form expression for the bit error probability (BEP) of convolutional codes is developed for coherent BPSK systems assuming that the interfering signal is expressed by Middleton's class A formula, which is a commonly used model of non-Gaussian interference. It is shown that the sum of identically distributed independent class A random variables becomes a new class A variable. Using this result, a simple expression for BEP is derived for Viterbi decoding with an unquantized soft decision. The validity of the expression is demonstrated by numerically simulating the BEP of a communication link. It is found that BEP after decoding under impulsive interference approaches that under Gaussian noise as the free distance of the code increases.**

Key words: Non-Gaussian interference, Convolutional code, Viterbi decoding, Bit error probability

I. Introduction

In order to develop appropriate limits of electromagnetic disturbances for protecting wireless services, it is necessary to evaluate the impact of non-Gaussian interference on the transmission quality of the disturbed wireless system. In particular, a method of evaluating the effect of impulsive interference is important because such interference may cause a severer degradation in the bit error probability (BEP) than Gaussian noise having the same power [1]. For the purpose of developing a disturbance limit, a simple closed expression is desirable, rather than an exact but complicated mathematical formula or the conduction of a time-consuming numerical simulation. For wireless systems not employing error correction coding, possible methods of evaluating the effect of impulsive interference have been developed such as amplitude probability distribution (APD) [2-3] and impulsive correction factors [4].

On the other hand, analyzing the BEP of coded systems is much more complicated than that of uncoded systems. In the case of Gaussian noise, mathematical expressions for BEP are known. However, a simple expression for the BEP of a coded system has not yet been obtained for most types of non-Gaussian interference mainly due to the complexity of the formulation.

On the basis of the above background, in this paper we derive a simple closed-form expression for the BEP of convolutional codes. For this purpose, we focus on the impulsive interference modelled by Middleton's class A formula [5]. It is known that the class A model can represent various types of non-Gaussian interference from highly impulsive to nearly Gaussian. The model has also been used for improving the performance of coding schemes in the presence of impulsive interference. In [6] and [7], the trellis-coded modulation and the turbo code under a class A interference were studied, respectively, and approximated forms of a metric were derived for improving the performance. In these papers, however, the BEP performance was obtained by conducting numerical simulations of the transmission scheme without deriving a mathematical expression.

For the purpose of discussing the BEP under class A interference, it is first shown that a sum of identically distributed independent class A variables becomes a new class A variable with a reduced impulsiveness. Using this result, an expression for the BEP under class A interference is derived for a BPSK system with convolutional coding decoded with an unquantized soft decision. On the basis of this expression and some results of numerical simulations, we discuss the effect of the free distance of the code on the BEP after decoding.

II. Probability Distribution of Sum of Independent Class A Random Variables

Middleton's class A formula gives the probability density function (pdf) of the in-phase (I) or quadrature (Q) component of an interfering signal. The pdf $p_Z(z)$ and complementary cumulative distribution function (ccdf) $f_Z(z)$ of the I or Q component, z, are given by

$$p_Z(z) = e^{-A}\sum_{m=0}^{\infty}\frac{A^m}{m!\sqrt{2\pi\sigma_m^2}}\exp\left(-\frac{z^2}{2\sigma_m^2}\right),\ \sigma^2{}_m \equiv \sigma^2\frac{m/A+\Gamma}{1+\Gamma},$$

$$\Gamma \equiv \sigma_G{}^2/\sigma_I{}^2,\ \sigma^2 = \sigma_G{}^2+\sigma_I{}^2, \tag{1}$$

and

$$f_Z(z) = \int_z^{\infty} p_Z(u)du, \tag{2}$$

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where A is the impulsive index, which is given by the average number of received pulses per unit time multiplied by the average pulse width. σ^2 denotes the total noise power. Γ represents the ratio of Gaussian noise power σ^2_G to impulsive noise power σ^2_I [1].

The shape of the ccdf curve of the interfering signal provides useful information for discussing the BEP under impulsive interference. In particular, the BEP of a binary phase shift keying (BPSK) system with coherent detection, on which we focus in the present paper, is expressed by the ccdf of the I component of the interfering signal. As shown in Fig. 1, the ccdf curve has a plateau if the interference is impulsive (i.e., $A < 1$ and $\Gamma A < 1$). The height and width of this plateau are approximately $(1\text{-}e^{-A})/2$ and $20\log(1+1/\Gamma A)^{1/2}$ in dB, respectively [8]. When the interfering signal is highly impulsive, the height and width can be further approximated by $A/2$ and $20\log(1/\Gamma A)^{1/2}$, respectively. If $A > 1$ or $\Gamma A > 1$, on the other hand, the ccdf curve does not have an apparent plateau. When $\Gamma A \gg 1$, the class A interference approaches purely Gaussian noise. Thus, we can roughly estimate the BEP of a BPSK system under class A noise with the given parameters, A and Γ, by sketching the ccdf curve.

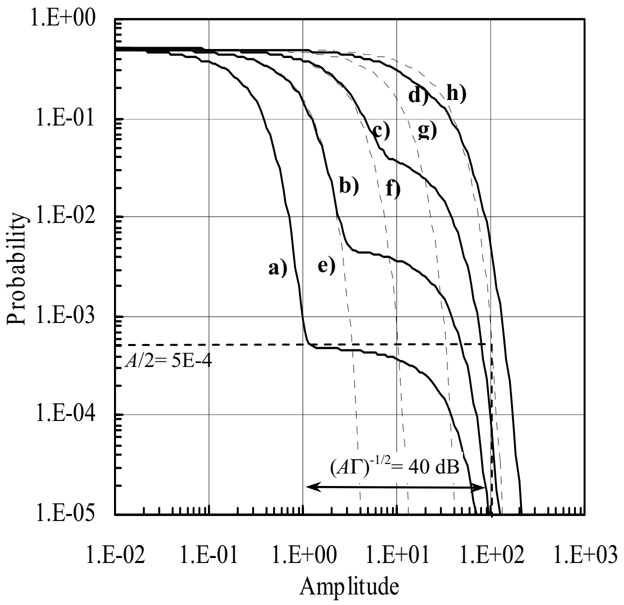

Fig. 1 Complementary cumulative distribution functions (ccdfs) of Middleton's class A noise and Gaussian noise.
Class A (solid lines): a) (Γ=0.1, A=0.001, variance σ^2=1), b) (0.1,0.01,10), c) (0.1,0.1,100), d) (0.1,1,1000)
Gaussian (dotted lines): e) σ^2=1, f) σ^2=10, g) σ^2=100, h) σ^2=1000.

In order to investigate the effect of the coding on the BEP, we consider the properties of a sum of class A random variables (RVs). Let us assume an RV given by the sum of K independent class A RVs as follows:

$$X \equiv \sum_{l=1}^{K} Z_l \, . \tag{3}$$

Note that all RVs, Z_l, are assumed to be identically distributed as given by (1) because we analyze the BEP under stationary class A interference in the subsequent sections.

It is known that a class A RV given by (1) can be understood as a Gaussian RV with zero mean and variance σ_m^2 conditioned on a Poisson RV, m [9]. Considering that the sum of independent Gaussian RVs is a Gaussian whose variance is the sum of those of the original RVs, and that sum of Poisson RVs is also a Poisson RV, we can show that the sum of class A RVs given by (3) is also a class A RV (the derivation is shown in the Appendix). The pdf of the sum, X, is given by

$$p_X(x) = e^{-AK} \sum_{M=0}^{\infty} \frac{(AK)^M}{M!\sqrt{2\pi\sigma_M^2 K}} \exp\left(-\frac{x^2}{2\sigma_M^2 K}\right), \tag{4}$$

$$\sigma_M{}^2 \equiv \sigma^2 \frac{\frac{M}{AK}+\Gamma}{1+\Gamma},$$

with the ccdf

$$f_X(x) = e^{-AK} \sum_{M=0}^{\infty} \frac{(AK)^M}{M!} Q\left(\frac{x}{\sqrt{\sigma_M^2 K}}\right), \tag{5}$$

where the Q-function is defined as $Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^{\infty} \exp\left(-\frac{x^2}{2}\right)$.

It should be noted that the new class A RV, X, has an impulsive index increased by the factor K, which means that the new class A RV is less impulsive than the original ones. When K tends to infinity, the pdf in (4) approaches the Gaussian pdf, which can also be understood by considering the central limit theorem.

III. BEP of Convolutional Codes Under Class A Interference

In the following, the BEP for a coherent BPSK system with convolutional codes is studied. We assume a Viterbi decoder with a metric optimized for AWGN that takes unquantized soft decisions. A commonly used upper bound of the BEP for convolutional codes is based on a union bound and the transfer function [10].

$$P_b < \frac{1}{k} \sum_{d=d_{\text{free}}}^{\infty} \beta_d P_2(d), \tag{6}$$

where k is defined by the code rate $R_c = k/n$ of the code, under the assumption that k and n have no common divider, and β_d denotes the number of incorrectly decoded information bits for each possible incorrect path that merges with the correct path. $P_2(d)$ and d_{free} represent the pairwise error probability and the minimum distance of the code, respectively. This bound is commonly truncated to the first term to derive an approximation for the BEP [10].

$$P_b \approx \frac{\beta_{d_{\text{free}}}}{k} P_2(d_{\text{free}}). \tag{7}$$

The pairwise error probability $P_2(d)$ for arbitrary interference is obtained as

$$\begin{aligned} P_2(d) &= \Pr\left[\sum_{l=1}^{d} \left(-\sqrt{E_c}\right) + \mathrm{Re}(n_l) \geq 0\right] \\ &= \Pr\left[X(d) \geq d\sqrt{E_c}\right] = \int_{d\sqrt{E_c}}^{\infty} p_X(u)\,du \end{aligned}, \tag{8}$$

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where E_c is the code bit energy related to the bit energy E_b as $E_c=R_cE_b$, n_l denotes the interference contribution, and $X(d)=\sum_{l=1}^{d}\mathrm{Re}(n_l)$. Here we can see that $P_2(d_{\mathrm{free}})$ is equal to the ccdf of $X(d)$. In particular, for AWGN, $P_2(d_{\mathrm{free}})=Q\left(\sqrt{2(E_b/N_0)R_c d_{\mathrm{free}}}\right)$. We assume that the interference can be modelled as stationary class A noise, i.e., $\mathrm{Re}(n_l)$ is identically class A distributed and that each n_l is independent. If the assumption of independence is not satisfied, a bit interleaver is necessary to randomize the order of the demodulated bit sequence. Under the above assumptions, we can make use of the result in (5) and derive $P_2(d)$ as

$$P_2(d) = e^{-Ad}\sum_{M=0}^{\infty}\frac{(Ad)^M}{M!}Q\left(\sqrt{\frac{R_cE_bd}{\tilde{\sigma}_M^2}}\right), \tag{9}$$

which is very similar to the BEP of an uncoded BPSK system [10].

$$P_{\mathrm{b\_uncode}} = e^{-A}\sum_{m=0}^{\infty}\frac{(A)^m}{m!}Q\left(\sqrt{\frac{E_b}{\tilde{\sigma}_m^2}}\right). \tag{10}$$

Using (7), the BEP of the coded BPSK system is obtained as

$$P_{\mathrm{b\_code}} \cong \frac{\beta_{d_{\mathrm{free}}}}{k}e^{-Ad_{\mathrm{free}}}\sum_{M=0}^{\infty}\frac{(Ad_{\mathrm{free}})^M}{M!}Q\left(\sqrt{\frac{R_cE_bd_{\mathrm{free}}}{\tilde{\sigma}_M^2}}\right). \tag{11}$$

Note that the variances $\tilde{\sigma}_M^2$ and $\tilde{\sigma}_m^2$ in (9) and (10) represents the interference power normalized by the bandwidth of the matched filter of the demodulator, B, and hence they have the unit of watt/Hz. Using the bit rate R and coding rate R_c, (10) and (11) can be rewritten in terms of C (carrier power) and I (interference power) on the substitutions of the following relations:

$$\begin{cases} E_b = C_{\mathrm{uncode}}/R_{\mathrm{uncode}}, \\ \tilde{\sigma}^2 = I_{\mathrm{uncode}}/B_{\mathrm{uncode}}, \\ \tilde{\sigma}_m^2 = \sigma_m^2/B_{\mathrm{uncode}}, \\ B_{uncode} = R_{uncode}, \end{cases} \quad \text{(uncoded system)} \tag{12}$$

$$\begin{cases} E_b = C_{\mathrm{code}}/R_{\mathrm{code}}, \\ \tilde{\sigma}^2 = I_{\mathrm{code}}/B_{\mathrm{code}}, \\ \tilde{\sigma}_M^2 = \sigma_M^2/B_{\mathrm{code}}, \\ B_{\mathrm{code}} = R_{\mathrm{code}}/R_c. \end{cases} \quad \text{(coded system)} \tag{13}$$

When the coded and uncoded BPSK systems have an identical bandwidth, i.e., $B_{\mathrm{code}}=B_{\mathrm{uncode}}=B_0$, the corresponding class A parameters, A, Γ, and $\tilde{\sigma}^2$, for the coded and uncoded systems are the same. Note that B_0 represents the reference bandwidth in which the class A noise parameters are defined. From (12) and (13), we have $R_{\mathrm{code}}=R_{\mathrm{uncode}}R_c$, $C_{\mathrm{code}}=C_{\mathrm{uncode}}R_c$, and $I_{\mathrm{code}}=I_{\mathrm{uncode}}$. Then the results of the substitutions are

$$P_{\mathrm{b\_coded}}=\frac{\beta_{d_{\mathrm{free}}}}{k}e^{-Ad_{\mathrm{free}}}\sum_{M=0}^{\infty}\frac{(Ad_{\mathrm{free}})^M}{M!}Q\left(\sqrt{d_{\mathrm{free}}\frac{C_{\mathrm{code}}}{I_{\mathrm{code}}}\frac{1+\Gamma}{(M/(Ad_{\mathrm{free}})+\Gamma)}}\right), \tag{14}$$

$$P_{\mathrm{b\_uncode}} = e^{-A}\sum_{m=0}^{\infty}\frac{(A)^m}{m!}Q\left(\sqrt{\frac{C_{\mathrm{uncode}}}{I_{\mathrm{uncode}}}\frac{1+\Gamma}{(m/A+\Gamma)}}\right). \tag{15}$$

(Constant bandwidth case: $C_{\mathrm{code}}=C_{\mathrm{uncode}}R_c$, $I_{\mathrm{code}}=I_{\mathrm{uncode}}$)

On the other hand, when the coded and uncoded systems have the same bit rate, i.e., $R_{\mathrm{code}}=R_{\mathrm{uncode}}$, we have $B_{\mathrm{uncode}}=B_{\mathrm{code}}R_c=B_0R_c$, $C_{\mathrm{uncode}}=C_{\mathrm{code}}$, and $I_{\mathrm{uncode}}=I_{\mathrm{code}}R_c$. Furthermore, note that the class A noise at the output of the matched filter of the uncoded system is less impulsive than that of the coded system because the uncoded system in this case has a narrower bandwidth as mentioned above. According to [11], the impulsive index of the uncoded system is increased to A/R_c, where A is defined in the reference bandwidth B_0. Thus, the BEP is given by

$$P_{\mathrm{b\_coded}}=\frac{\beta_{d_{\mathrm{free}}}}{k}e^{-Ad_{\mathrm{free}}}\sum_{M=0}^{\infty}\frac{(Ad_{\mathrm{free}})^M}{M!}Q\left(\sqrt{d_{\mathrm{free}}\frac{C_{\mathrm{code}}}{I_{\mathrm{code}}}\frac{1+\Gamma}{(M/(Ad_{\mathrm{free}})+\Gamma)}}\right), \tag{16}$$

$$P_{\mathrm{b\_uncode}} = e^{-A/R_c}\sum_{m=0}^{\infty}\frac{(A/R_c)^m}{m!}Q\left(\sqrt{\frac{C_{\mathrm{uncode}}}{I_{\mathrm{uncode}}}\frac{1+\Gamma}{(m/(A/R_c)+\Gamma)}}\right). \tag{17}$$

(Constant bit rate case: $C_{\mathrm{uncode}}=C_{\mathrm{code}}$, $I_{\mathrm{uncode}}=I_{\mathrm{code}}R_c$)

It is known that each infinite summation in (14)-(17) can be well approximated by the sum of the first three terms (m or M = 0, 1, and 2) for practical applications [8].

IV. Discussion

Numerical simulations were conducted in order to examine the accuracy of the expression for BEP (11). In Fig. 2, BEPs obtained by the simulations for three different convolutional codes are shown as a function of $E_b/\tilde{\sigma}^2$. The codes have the code rate R_c=1/2 and the generator polynomials A=[5 7], B=[133 171], and C=[4335 5723], with (d_{free}, $\beta(d_{\mathrm{free}})$)=(5,1), (10,36), and (15, 76), respectively. In all cases, class A interference with A=0.1 and Γ=0.1 is assumed. The results demonstrate that (14) provides a good approximation to the BEP. It is also found that the BEP curves for class A interference become closer to those for AWGN as the free distance of the code increases.

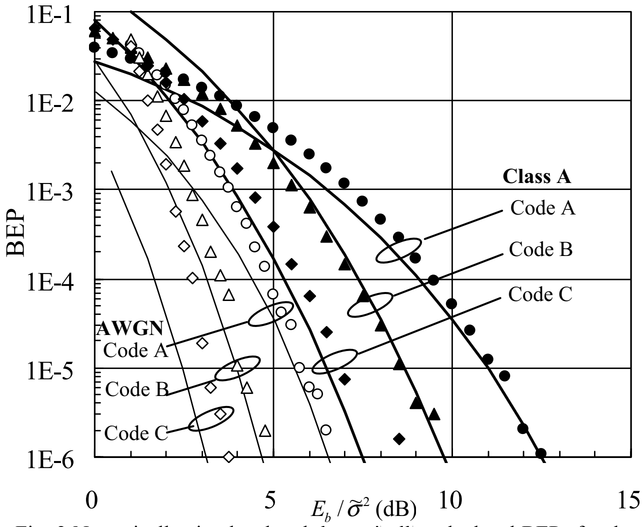

Fig. 2 Numerically simulated and theoretically calculated BEPs for three different convolutional codes. Marks: simulated, solid lines: calculated. Thick lines: class A interference (A=0.1, Γ=0.1), thin lines: AWGN. Code A: [5 7], d_{free}=5, $\beta(d_{\mathrm{free}})$=1. Code B: [133 171], d_{free}=10, $\beta(d_{\mathrm{free}})$=36. Code C: [4335 5723], d_{free}=15, $\beta(d_{\mathrm{free}})$=76.

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The last statement can also be explained by studying the expression of BEP. As is shown in (14), the BEP can be approximated directly by the ccdf of $X(d_{\text{free}})$ in the case of coherently detected BPSK. To analyze the BEP, we recall the fact that the plateau width of the ccdf of class A interference in an impulsive case can be approximated as 10log(1+1/ΓA) in dB. As ΓA increases, the plateau width approaches zero, and the ccdf approaches that for a Gaussian noise as mentioned in section II. In the following discussions, we consider the values of $E_b/\tilde{\sigma}^2$, γ_{nG} and γ_{AWGN}, necessary to achieve a certain BEP in class A interference and AWGN, respectively. By comparing the class A ccdf curve in Fig. 1 with the Gaussian ccdf that has the same variance (for example, curves **a** and **e**), we can conclude the following.

1) In the probability range above the plateau, a class A ccdf generally has lower values than the corresponding Gaussian ccdf, which means that Gaussian noise causes greater degradation than class A interference.
2) In contrast, the BEP in class A interference below the plateau is higher than that in AWGN. This means that for a certain BEP, $\gamma_{\text{nG}} - \gamma_{\text{AWGN}} \geq 0$.
3) However, the difference γ_{nG} - γ_{AWGN} approaches zero as d_{free} increases (i.e., the impulsive index $d_{\text{free}}A$ increases),

$$\gamma_{\text{nG}} - \gamma_{\text{AWGN}} \Big|_{d_{\text{free}}=d_1} > \gamma_{\text{nG}} - \gamma_{\text{AWGN}} \Big|_{d_{\text{free}}=d_2} > \ldots > \gamma_{\text{nG}} - \gamma_{\text{AWGN}} \Big|_{d_{\text{free}}=d_n} \to 0, \quad d_n \to \infty \quad (18)$$

since $\gamma_{\text{nG}} \to \gamma_{\text{AWGN}}$ as $d_n \to \infty$, where $d_1 < d_2 < \ldots < d_n$. This can also be explained by the central limit theorem. For these reasons, the BEP of coherent BPSK systems caused by class A interference will approach that of AWGN as the free distance of the code increases. However, it should be noted that the rate at which γ_{nG} approaches γ_{AWGN} depends on the parameters of the class A noise, A and Γ.

V. Conclusion

A simple closed-form expression for the BEP of convolutional codes is developed for the coherent detection of BPSK under class A interference. On the basis of the expression, we discussed the effects of the class A parameters and the free distance of the code on the BEP after decoding.

Appendix

The characteristic function (CF) of each class A RV Z_l is [12]

$$F_{Z_l}(\xi) = e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} \exp\left(-\frac{\sigma_m^2 \xi^2}{2}\right). \quad (A1)$$

The CF of X can be derived by multiplying the CF of each class A variable as follows:

$$F_X(\xi) = \prod_{l=1}^{K} F_{Z_l}(\xi)$$

$$= e^{-AK} \underbrace{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\cdots\sum_{p=0}^{\infty}}_{K} \frac{A^{m+n+\ldots+p}}{m!\,n!\ldots p!} \exp\left(-\frac{\left(\sigma_m^2+\sigma_n^2+\ldots+\sigma_p^2\right)\xi^2}{2}\right). \quad (A2)$$

By letting M=m+n+...+p, (A2) can be rewritten as

$$F_X(\xi) = e^{-AK} \sum_{M=0}^{\infty} A^M \exp\left(-\frac{\xi^2 K\sigma^2\Gamma}{2(1+\Gamma)}\right) \sum_{\substack{m+\ldots+p=M \\ m,\ldots,p\geq 0}} \frac{1}{m!\,n!\ldots p!} E,$$

$$E \equiv \exp\left(\frac{-\xi^2\sigma^2}{2A(1+\Gamma)}\right)^m \exp\left(\frac{-\xi^2\sigma^2}{2A(1+\Gamma)}\right)^n \ldots \exp\left(\frac{-\xi^2\sigma^2}{2A(1+\Gamma)}\right)^p. \quad (A3)$$

Note that the inner sum in (A3) is taken for all sets of nonnegative integers (m, n, ... , p) that satisfy m+n+...+p=M. Using the multinomial theorem, (A3) can be modified as

$$F_X(\xi) = e^{-AK} \sum_{M=0}^{\infty} \frac{(AK)^M}{M!} \exp\left(-\frac{\xi^2 K\sigma_M^2}{2}\right), \quad (A4)$$

$$\sigma_M^2 \equiv \sigma^2 \frac{\frac{M}{AK}+\Gamma}{1+\Gamma},$$

which yields the probability distribution of a class A RV given by (4) and (5) that has a variance and impulsive index changed from σ^2 to $K\sigma^2$ and from A to KA, respectively, compared with those of the original distribution.

References

[1] K. Wiklundh, "A comparison between the impact of some interfering signals on an MSK receiver," *1999 IEEE Military Comm. Conf.*, pp. 51-55, Atlantic City, Nov. 1999.

[2] K. Wiklundh, "Relation between the amplitude probability distribution of an interfering signal and its impact on digital radio services," *IEEE Trans. on Electromagnetic Compatibility*, vol. 48, no. 3, pp. 537-544, Aug. 2006.

[3] Y. Matsumoto, "On the relation between the amplitude probability distribution of a noise and bit error probability," *IEEE Trans. on Electromagnetic Compatibility*, vol. 49, no. 4, pp. 940-941, Nov. 2007.

[4] P. Stenumgaard, "A simple impulsive correction factor for control of electromagnetic interference in dynamic wireless applications," *IEEE Communications Letters*, vol. 10, no. 3, pp. 147-149, Mar. 2006.

[5] D. Middleton, "Statistical-physical models of electromagnetic interference," *IEEE Trans. on Electromagnetic Compatibility*, vol. 19, no. 3, pp. 106-127, Aug. 1977.

[6] S. Miyamoto, M. Katayama, and N. Morinaga, "Optimum detection and design of TCM signals under impulsive noise environment," *IEEE Int. Conf. on System Engineering*, pp. 473-478, Kobe, Sept. 1992.

[7] D. Umehara and Y. Morihiro, "Turbo decoding in impulsive noise environment," *IEEE Globecom 2004*, pp. 194-198, Dallas, Dec. 2004.

[8] L. A. Berry, "Understanding Middleton's canonical formula for class A noise," *IEEE Trans. on Electromagnetic Compatibility,* vol. 23, no. 4, pp. 337-344, Nov. 1981.

[9] C. Tepedelenlioglu and P. Gao, "On diversity reception over fading channels with impulsive noise," *IEEE Trans. on Vehicular Technology*, vol. 54, no. 6, pp. 2037-2046, Nov. 2005.

[10] J. G. Proakis, *Digital Communications*, 3rd ed., McGraw-Hill International Editions, 1995.

[11] Y. Matsumoto, K. Gotoh, and K. Wiklundh, "Band-limitation effect on statistical properties of class-A interference," *EMC Europe 2008*, pp. 715-719, Hamburg, Sept. 2008.

[12] A. D. Spaulding and D. Middleton, "Optimum reception in an impulsive interference environment – Part I: Coherent detection," *IEEE Trans. on Communications,* vol. 25, no. 9, pp. 910-923, Sept. 1977.

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