Some Problems of Diagnostics of Inhomogeneous Media

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Abstract — **We explore possibilities of developing an asymptotic approach based on the double weighted Fourier transform (DWFT), proposed earlier to solve problems of wave propagation in inhomogeneous media. Two types of problems are considered: the generalization of the phase screen method to extended inhomogeneous media and wave reflection from a layer with random irregularities. We determine conditions of validity of the phase screen model and criteria for choice of position of the screen in diagnostics of an inhomogeneous medium with advanced resolution.**

I. Introduction

Success in any inhomogeneous media diagnostic technique depends largely on a space-time sounding signal model in use which in its turn is determined by a method for describing wave propagation. Wave propagation in media with large-scale irregularities typical sizes of which exceed wavelength are often described using the geometrical optics (GO) approximation [1–4]. Limitations of corresponding diagnostic techniques are due to the fact that the GO approximation ignores diffraction effects and caustics arising during wave propagation in inhomogeneous media. The method of smooth perturbations (Rytov approximation) accounts for Fresnel diffraction effects [2–4]. However, this method is valid only without multipath propagation and caustics and therefore is unsuitable for describing strong amplitude fluctuations. Irregularities of a medium are fairly often located in a bounded region far from an observer. In this case, one often uses the field continuation from a plane in the vicinity of the irregularities to the desired region with the aid of Green's formula [5–6]. When an inhomogeneous layer is so small that we can neglect not only diffraction effects and caustics but also amplitude variations appearing within this layer, this approach transforms to the phase screen method [2, 3, 7]. The widespread acceptance of the phase screen method both for analyzing wave propagation and for developing diagnostic techniques is attributed to the fact that despite its simplicity it accounts for diffraction effects and strong amplitude fluctuations associated with appearance of caustics. However, when employing the phase screen method to describe wave propagation in an extended inhomogeneous medium, we face problems of specifying virtual screen coordinates [7]. Previously proposed [8–11] DWFT-based description of the field in an inhomogeneous medium provides solution to these problems. The spatial signal processing based on the inverse DWFT enables us to advance resolution of inhomogeneous media diagnostic tools [9–11].

However, the use of DWFT for direct and inverse problems of wave propagation in inhomogeneous media is often hampered by the necessity to integrate in two planes. In particular, in tropospheric and ionospheric investigations we can integrate measurement data in coordinates of only one plane, say a receiving plane.

Another problem of using the DWFT method is the small-angle approximation exploited in this method which forbids its application to reflectometry problems.

Here we will consider the DWFT method modifications permitting the use of it both without possibility of integrating over two planes and under wave reflection from an inhomogeneous medium (plasma).

II. The Double Weighted Fourier Transform (DWFT) for a Remote Irregularity

Let a time-dependent source $e^{-i\omega t}$ and a receiver be at the respective points $\mathbf{r}_0=(z_0,x_0,y_0)=(z_0,\boldsymbol{\rho}_0)$ and $\mathbf{r}=(z_t,x,y)=(z_t,\boldsymbol{\rho})$, where $\boldsymbol{\rho}_0=(x_0,y_0)$ and $\boldsymbol{\rho}=(x,y)$ are two-dimensional vectors in planes $z=z_0$ and $z=z_t$. Between the source and the receiver is an inhomogeneous medium. Scattering, in the absence of reflection in a large-scale inhomogeneous medium, is largely forward. In this case, in solving wave equation

$$\frac{\partial^2}{\partial \mathbf{r}^2}U(\mathbf{r},\mathbf{r}_0)+k^2\varepsilon(\mathbf{r})U(\mathbf{r},\mathbf{r}_0)=\delta(\mathbf{r}-\mathbf{r}_0) \quad (1)$$

we can use a small-angle (paraxial) approximation and reduce it to the parabolic equation

$$2ik\frac{\partial U_1}{\partial z}+\frac{\partial^2 U_1}{\partial \rho^2}+k^2\left[\varepsilon(\mathbf{r})-1\right]U_1=0\,, \quad (2)$$

$$U_1\big|_{z=z_0}=\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0) \quad (3)$$

for

$$U_1(\mathbf{r},\mathbf{r}_0) = U(\mathbf{r},\mathbf{r}_0)\exp\{-ik[z-z_0]\} \qquad (4)$$

According to [9-11] we solve (2)-(3) with DWFT method and write down an expression for the field of the point source in the form:

$$U(\boldsymbol{\rho},\boldsymbol{\rho}_0) = -\frac{A_0k^2}{4\pi^3Z^3}\exp\{ik[Z+(\boldsymbol{\rho}+\boldsymbol{\rho}_0)^2/(2Z)]\}$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\xi d^2\xi_0 \exp\{ik[2(\boldsymbol{\xi}\boldsymbol{\xi}_0 - \boldsymbol{\xi}\boldsymbol{\rho}_0 - \boldsymbol{\xi}_0\boldsymbol{\rho})/Z + \varphi(\boldsymbol{\xi},\boldsymbol{\xi}_0)]\}, \qquad (5)$$

where

$$\varphi(\boldsymbol{\xi},\boldsymbol{\xi}_0) = \frac{1}{2}\int_{z_0}^{z_t}\tilde{\varepsilon}\left[\boldsymbol{\xi}\frac{z'-z_0}{Z}+\boldsymbol{\xi}_0\frac{z_t-z'}{Z},z'\right]dz', \qquad (6)$$

$Z = z_t - z_0$ is the distance between planes with the source and the receiver; A_0 is the incident spherical wave amplitude; $k=\omega/c$, $\omega = 2\pi f$ is the radiation frequency; c is the velocity of light in free space; $\tilde{\varepsilon}(\mathbf{r}) = \varepsilon(\mathbf{r}) - \bar{\varepsilon}(\mathbf{r})$ is the permittivity variation $\varepsilon(\mathbf{r})$ induced by irregularities; $\bar{\varepsilon}(\mathbf{r})$ is the background irregularity permittivity which in this case is assumed to be homogeneous with $\bar{\varepsilon}(\mathbf{r}) = 1$.

If l_ε is the minimum size of irregularities and condition

$$a_{fr}(z_0,z_t,z') = \sqrt{(z'-z_0)(z_t-z')/(kZ)} \ll l_\varepsilon \qquad (7)$$

is fulfilled for all z' within the inhomogeneous region, the fourfold integral in (5) is calculated by the stationary phase method, and (5) reduces to the GO approximation. Given weak phase fluctuations, (5) yields results of the Rytov method [9], and given small extent of the inhomogeneous medium in z (5) gives results of the phase screen method [11]. Thus, representation (5) accounts for both Fresnel diffraction effects during weak phase fluctuations and strong fluctuations associated with multipath propagation and caustics.

Let the irregularity under study be in the vicinity of $z = z_m$. Come to new coordinates $\boldsymbol{\rho}_b$, $\mathbf{p}_s$ in (5):

$$\boldsymbol{\xi}_0 = \boldsymbol{\rho}_b + \mathbf{p}_s(z_b - z_0),\quad \boldsymbol{\xi} = \boldsymbol{\rho}_b - \mathbf{p}_s(z_t - z_b), \qquad (8)$$

where $z = z_b$ is a virtual plane (screen) in the vicinity of $z = z_m$. If condition

$$ka_{fr}(z_0,z_t,z_b) \gg |z_b - z'|/l_\varepsilon \qquad (9)$$

is met, the integral in $\mathbf{p}_s$ is calculated by the stationary phase method, and we obtain

$$U(\boldsymbol{\rho},\boldsymbol{\rho}_0) = \frac{iA_0k}{8\pi^2(z_t-z_b)(z_b-z_0)}\int_{-\infty}^{\infty}\frac{d^2\rho_b}{\sqrt{\Delta_s}}$$
$$\times\exp\left[ik\left(Z+\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}_b)^2}{2(z_t-z_b)}+\frac{(\boldsymbol{\rho}_b-\boldsymbol{\rho}_0)^2}{2(z_b-z_0)}+\varphi_b(\boldsymbol{\rho}_b,\mathbf{p}_{sc})\right)\right], \qquad (10)$$

where

$$\Delta_s = 1-\frac{1}{8}\frac{Z}{(z_t-z_b)(z_b-z_0)}$$
$$\times\int_{z_0}^{z_t}(z_b-z')^2\frac{\partial^2}{\partial\rho'^2}\tilde{\varepsilon}[\boldsymbol{\rho}',z']dz', \qquad (11)$$

$\boldsymbol{\rho}' = \boldsymbol{\rho}_b + \mathbf{p}_{sc}(z_b - z')$.

Condition (9) is readily fulfilled if the region with irregularities is located far from a source and an observer. The second validity criterion of (10) is the absence of focusing in partial waves. In view of (11) it takes the form

$$\left|\frac{Z}{(z_t-z_b)(z_b-z_0)}\int_{z_0}^{z_t}(z_b-z')^2\frac{\partial^2\tilde{\varepsilon}[\boldsymbol{\rho}',z']}{\partial\rho'^2}dz'\right| < 1. \qquad (12)$$

Given small extent of the inhomogeneous medium, conditions (9), (12) are weaker than condition of validity of the GO approximation (7); hence (10) may be used for describing diffraction effects. At small thickness of the region with irregularities, (10) yields the phase screen approximation. So, (10) taking into consideration variations of both a phase in the inhomogeneous region and partial wave amplitudes may be interpreted to be the amplitude-phase screen approximation accounting for propagation effects within the inhomogeneous layer.

III. FRESNEL INVERSION

Consider action of an inverse operator as Fresnel field transform $U(\boldsymbol{\rho},\boldsymbol{\rho}_0)$:

$$\hat{U}(\boldsymbol{\rho}^*,\boldsymbol{\rho}_0) = \hat{L}_b[U(\boldsymbol{\rho},\boldsymbol{\rho}_0)]$$
$$= C_0\int_{-\infty}^{\infty} d^2\rho U(\boldsymbol{\rho},\boldsymbol{\rho}_0)\exp\left\{-ik\frac{(\boldsymbol{\rho}^*-\boldsymbol{\rho})^2}{2(z_t-z_b)}-ikZ\right\}. \qquad (13)$$

Fresnel inversion (13) is usually used to convert measurement data from a measuring plane to field values on the plane of the virtual screen $z = z_b$, situated on the way out of the inhomogeneous medium provided that there are no irregularities between these planes. To analyze characteristics of this processing, we substitute (5) in (13) and obtain

$$\hat{U}(\boldsymbol{\rho}^*,\boldsymbol{\rho}_0)=\frac{iC_0A_0k(z_t-z_b)}{2\pi^2(z_b-z_0)}\exp\left[ik\frac{(\boldsymbol{\rho}^*+\boldsymbol{\rho}_0)^2}{2(z_b-z_0)}\right]$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^2\rho_b d^2p_s\exp\left\{ik\left[\frac{2\rho_b^2}{z_b-z_0}+2\mathbf{p}_s(\boldsymbol{\rho}_b-\boldsymbol{\rho}^*)\right.\right. \quad (14)$$
$$\left.\left.-\frac{2\boldsymbol{\rho}_b(\boldsymbol{\rho}_0+\boldsymbol{\rho}^*)}{z_b-z_0}+\varphi_b(\boldsymbol{\rho}_b,\mathbf{p}_s)\right]\right\}.$$

If conditions

$$\left|a_{fr}(z_0,z_b,z')\right|=\left|\sqrt{\frac{(z'-z_0)(z_b-z')}{k(z_b-z_0)}}\right|<<l_\varepsilon, \quad (15)$$

$$\left|\int_{z_0}^{z_t}\frac{(z_b-z')(z'-z_0)}{(z_b-z_0)}\frac{\partial^2}{\partial\rho^2}\tilde{\varepsilon}\left[\mathbf{S}_0(z'),z'\right]dz'\right|<1 \quad (16)$$

are met, integral (14) may be calculated using the stationary phase method and give

$$\hat{U}(\boldsymbol{\rho}^*,\boldsymbol{\rho}_0)=\frac{iC_0A_0(z_t-z_b)}{2k(z_b-z_0)}$$
$$\times\exp\left\{ik\frac{(\boldsymbol{\rho}^*-\boldsymbol{\rho}_0)^2}{2(z_b-z_0)}+\frac{ik}{2}\int_{z_0}^{z_t}\tilde{\varepsilon}\left[\mathbf{S}_0(z'),z'\right]dz'\right\} \quad (17)$$
$$\times\left\{1+\frac{1}{2}\int_{z_0}^{z_t}\frac{(z_b-z')(z'-z_0)}{(z_b-z_0)}\frac{\partial^2}{\partial\rho^2}\tilde{\varepsilon}\left[\mathbf{S}_0(z'),z'\right]dz'\right\}^{-1/2},$$

where

$$\mathbf{S}_0(z')=\boldsymbol{\rho}^*\frac{z'-z_0}{z_b-z_0}+\boldsymbol{\rho}_0\frac{z_b-z'}{z_b-z_0}. \quad (18)$$

The virtual screen is normally set up between an irregularity and a receiver, i.e. when

$$z_t>z_b>z'. \quad (19)$$

In this case, $a_{fr}^2(z_0,z_b,z')>0$ coincides with the squared Fresnel radius in the plane of the irregularity $z=z'$ if the source and the observer are in $z=z_0$ and $z=z_b$ respectively. The upper limit of integration in (17) may be changed for z_b, and the result of processing (13) appears to be proportional to the GO field approximation on the way out of the layer in the plane $z=z_b$. Thus, algorithm (13) converts observations from the diffraction region, where condition (7) is ruled out, to the region close to the irregularity with fulfilled GO approximation conditions (15)-(16). Equation (17) shows the absence of multipath and diffraction effects in this region. This brings about an improvement in accuracy and resolution of measurements of linear integral (6) with the help of (13). The same strategy is realized in Fresnel inversion methods (e.g., see [6, 12]). However, in contrast to these methods, formulas (15)-(16) suggest possibility of using inversion (13) even if condition (19) is not fulfilled when $a_{fr}^2(z_0,z_b,z')$ becomes zero or negative; i.e. the screen is put in an inhomogeneous region or between a source and an inhomogeneous region.

So, linear integral (6) may be derived from measurement data by taking diffraction effects into account and using single weighted Fourier transform (13) if conditions (15)-(16) are met. Equations (16)-(17) imply the rule of choice of a virtual screen coordinate z_b, i.e. it should be chosen from the minimum condition of amplitude variation of wave (17) after spatial processing (13).

IV. Wave Reflection from a Layer with Random Irregularities

To solve the problem of wave reflection from a layer with random irregularities, we will apply the Fock proper-time method [13] in which (1) reduces to the parabolic equation through transformation

$$U(\mathbf{r},\mathbf{r}_0)=-i/(2k)\int_0^\infty V(\mathbf{r},\mathbf{r}_0,\tau)d\tau. \quad (20)$$

It is easy to show that $V(\mathbf{r},\mathbf{r}_0,\tau)$ satisfies the following equation

$$2ik\frac{\partial V}{\partial\tau}+\frac{\partial^2}{\partial\mathbf{r}^2}V+k^2\varepsilon(\mathbf{r})V=0 \quad (21)$$

with initial condition

$$V\big|_{\tau=0}=\delta(\mathbf{r},\mathbf{r}_0). \quad (22)$$

As an example of application of this approach let us consider the case when the background medium is a layer with permittivity

$$\varepsilon(z)=\bar{\varepsilon}(z)=1-\varepsilon'z. \quad (23)$$

Without irregularities, solving (21)-(23) by the geometrical optics method yields:

$$U(\mathbf{r},\mathbf{r}_0)=\frac{e^{\frac{i3\pi}{4}}}{4\pi}\left(\frac{k}{2\pi}\right)^{1/2}\int_0^\infty d\tau\tau^{-3/2}\exp\{ik[\tau/2$$
$$+(\mathbf{r}-\mathbf{r}_0)^2/(2\tau)-\varepsilon'\tau(z_0+z)/4-\varepsilon'^2\tau^3/96]\}. \quad (24)$$

Formula (24) is precisely the strict solution [14]. So, this approach provides a correct description of the field in the vicinity of the turning point. To take into account the Fresnel diffraction by random irregularities located in the vicinity of the turning point, we employ the DWFT method in solving (21) and get

$$U(\mathbf{r},\mathbf{r}_0)=\frac{4}{k}e^{\frac{i3\pi}{4}}\left(\frac{k}{2\pi}\right)^{9/2} \times\int_0^{\infty}\frac{d\tau}{\tau^{9/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^3\xi d^3\xi_0 \exp\left\{ik\left[\frac{\tau}{2}+\frac{(\mathbf{r}-\mathbf{r}_0)^2}{2\tau} +\frac{2}{\tau}(\xi_0-\mathbf{r}_0)(\xi-\mathbf{r})-\frac{\varepsilon'\tau}{4}(z_0+z)-\frac{\varepsilon'^2\tau^3}{96}+\tilde{\Phi}\right]\right\}, \tag{25}$$

where

$$\tilde{\Phi}=\tilde{\Phi}(\xi,\xi_0,\tau)=0.5\int_0^{\tau}d\tau'\tilde{\varepsilon}\left[\xi_0(1-\tau'/\tau) +\xi\tau'/\tau+\mathbf{e}_z\varepsilon'(\tau'-\tau)\tau'/4\right]. \tag{26}$$

At $\varepsilon'=0$, (25) gives results obtained in [15]

$$U(\mathbf{r},\mathbf{r}_0)=\frac{4}{k}e^{\frac{i3\pi}{4}}\left(\frac{k}{2\pi}\right)^{9/2}\int_0^{\infty}\frac{d\tau}{\tau^{9/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^3\xi d^3\xi_0 \exp\left\{ik\left[\frac{\tau}{2} +\frac{(\mathbf{r}-\mathbf{r}_0)^2}{2\tau}+2\frac{(\xi_0-\mathbf{r}_0)(\xi-\mathbf{r})}{\tau}+\tilde{\Phi}_1(\xi,\xi_0,\tau)\right]\right\}, \tag{27}$$

where

$$\tilde{\Phi}_1(\xi,\xi_0,\tau)=1/2\int_0^{\tau}\tilde{\varepsilon}\left[\xi_0(1-\tau'/\tau)+\xi\tau'\right]d\tau'. \tag{28}$$

As is shown in [15], at $k\left|\tilde{\Phi}_1\right|<<1$ (27) takes the form of the Born approximation, hence our solution incorporates diffraction effects and back scattering. In the small-angle approximation, (27) yields DWFT results (5).

V. Conclusion

If an irregularity is far from a source and an observer, we can use single integration instead of double one inherent in the DWFT method. This brings about an improvement in the measurement resolution through spatial processing. Contrary to the usual Fresnel inversion, the proposed approach by monitoring changes of amplitude variations lets us set up a virtual screen quasi-optimally for diagnostics of irregularities in phase measurements. A similar spatial processing is also possible in reflectometry of an inhomogeneous medium. The proposed sounding signal field description methods and ways of its quasi-optimal processing can be applied to diagnostics of ionospheric and thermonuclear fusion plasmas.

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