From Galilean covariance to Maxwell equations: A thermodynamic insight to electromagnetism

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***Abstract*— Galilean electromagnetism is derived from a thermodynamic approach. Attention is paid on the various regimes allowed by the quasi-static limit. It is emphasized that an abrupt transition exists between the QS-magnetic and the QS-electric regimes for which different gauge conditions on the potentials should be considered. Besides, due to its quadratic formulation, the approach is shown to be intrinsically multi-scaled and suitable for energy-based homogenization.**

***Index Terms*— electromagnetism, quasi-static regimes, power integrity, gauges, homogenization, finite element method.**

I. Introduction

This paper is an attempt to promote a physics-oriented framework dedicated to the consolidation (i) in space, of all the scales involved in the conversion process of the electromagnetic energy; (ii) in time, over all the life cycle of the electrical supply-chain. Whereas the former addresses some issues to progress on materials and design, the latter appears related to network management and EMC considerations.

The problem is here addressed within the framework of the Galilean electromagnetism introduced in [1].

In the following, a variational approach of electromagnetism is proposed. Hence, the two regimes occurring within the quasi-static limit are discussed from the covariance viewpoint and the transition in between is explained as a change in the gauge condition. Besides, homogenization procedure is investigated from spatial averaging techniques. The variational approach is shown to be robust providing a suitable multi-scale theory for power management.

II. Variational formulation

Classically, thermodynamic approaches of electromagnetism do not consider any extension towards time-varying regimes [2][3][4]. Whereas some improvements are summarized in [5] for steady states regimes, no general contribution is available for transient. Denoting, as a general rule in this paper, variational parameters or functionals thanks to *italic* fonts whereas roman ones specify their values at the minimum, the magnetodynamic behavior of any electrical system is derived from the functional [6]:

$$-\mathrm{P}_{\mathrm{mech}} - \frac{\mathrm{dG}}{\mathrm{d}t} = \min_{\boldsymbol{H},\boldsymbol{E}} \left(P_{\mathrm{Joule}} + \frac{\mathrm{d}}{\mathrm{d}t} \int \left(\mathbf{B} \cdot \boldsymbol{H} + \mathbf{D} \cdot \boldsymbol{E} \right) \mathrm{d}^3 r \right) \tag{1}$$

where the functional in the RHS exhibits:

- the magnetic field $\boldsymbol{H}$ related to the free current density $\boldsymbol{J}$ and the displacement density $\frac{\partial \boldsymbol{D}}{\partial t}$ according to the Maxwell-Ampere equation:

$$\mathbf{curl} \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{2}$$

While the quasi-static approximation enforces $\boldsymbol{D} \equiv 0$ in conductors (*i.e.* no free charge), the charge conservation at the various interfaces enforces the continuity equation:

$$\mathbf{n} \times \left([\boldsymbol{H}] - \mathbf{V} \times [\boldsymbol{D}] \right) = 0 \tag{3}$$

where $\mathbf{V}$ is the velocity of the interface in the rest frame, $\mathbf{n}$ is the unit vector oriented by the interface, and $[\cdot]$ denotes the field discontinuity occurring thereon;

- the Joule losses monitored in conductors, basically expresed as:

$$P_{\mathrm{Joule}} = \int_C \sigma^{-1} \left(\mathbf{curl} \boldsymbol{H} \right)^2 \mathrm{d}^3 r \tag{4}$$

This term is quadratic to respect invariance of losses with inversion of time inversion ($\sigma^{-1} > 0$ is the resistivity);

- the variation with time of the electromagnetic energy coupling the field with the generator supplied by the net current I and the mass at the potential V_0;
- the magnetic $\mathbf{B}(\boldsymbol{h})$ and electrostatic $\mathbf{D}(\boldsymbol{e})$ behavior laws derived from the thermostatistic equilibrium of the Gibbs potential:

$$\begin{aligned} G(\mathrm{I}, \mathrm{V}_0) &= G_{\mathrm{m}} + G_{\mathrm{e}} \\ &= \int \left(\int_0^{\boldsymbol{H}} (-\boldsymbol{B}) \cdot \delta \boldsymbol{h} + \int_0^{\boldsymbol{E}} (-\boldsymbol{D}) \cdot \delta \boldsymbol{e} \right) \mathrm{d}^3 r \end{aligned} \tag{5}$$

where the flux density $\mathbf{B}$ and the electric displacement $\mathbf{D}$ are divergence-free to ensure that Gibbs' free-energy G is a state-function.

Galilean covariance states that stationary conditions expressed from (1) do adopt an independent form from the Galilean frame where the time-derivation is performed. Introducing $\mathbf{V}'$ as the relative velocity of the frame ($'$) in (1), some calculations on the convective derivative of the electromagnetic energy coupling yield the transformation law for the electric field:

$$\mathbf{E}' = \mathbf{E} + \mathbf{V}' \times \mathbf{B} \tag{6}$$

whereas the flux density and the electric displacement are retained:

$$\mathbf{B}' = \mathbf{B} \tag{7}$$

$$\mathbf{D}' = \mathbf{D} \tag{8}$$

Hence, extending the electric field in a moving conductor (with the velocity $\mathbf{V}$) according to Ohm's law with motion:

$$\mathbf{E} = \sigma^{-1}\mathbf{J} - \mathbf{V} \times \mathbf{B} \tag{9}$$

Maxwell-Faraday's equation:

$$\mathbf{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10}$$

and its subsequent continuity equation:

$$\mathbf{n} \times ([\mathbf{E}] + \mathbf{V} \times [\mathbf{B}]) = 0 \tag{11}$$

may be viewed as acting *locally* to check *globally* the tendency towards reversibility expressed by (1) [7]. This striking property provides a thermodynamic-oriented insight of the variational theory of electromagnetism [8]. Hence, the functional (1) balances the variations with time of the co-energy $(-\mathrm{G})$ and the mechanical power received by the field from the actuator $(-\mathrm{P}_{\mathrm{mech}})$.

It should be noticed that Maxwell-Faraday equation (10) provides divergence-free flux density at any time, which is a mandatory condition to derive (6) in its usual form. Similarly, the transformation law of the magnetic field reads:

$$\mathbf{H}' = \mathbf{H} - \mathbf{V}' \times \mathbf{D} \tag{12}$$

according to (3).

In order to consider sub-systems for design purpose, it is convenient to introduce the electrical power:

$$P_{\mathrm{elec}}(\Omega) = -\oint_{\partial\Omega} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathbf{n}\; \mathrm{d}^2 r \tag{13}$$

After some calculations taking into account possible interfacial discontinuities of $\boldsymbol{E}$ and $\boldsymbol{H}$ occurring at the interfaces $\partial\Omega_i \subset \Omega$, it follows, after some transformations involving convective derivation:

$$\begin{aligned} P_{\mathrm{elec}}(\Omega) = &-\int_{\Omega} \left(\mathbf{curl}\,\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t}\right) \cdot \boldsymbol{H}\mathrm{d}^3 r \\ &+ \int_{\Omega} \left(\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{J} - \frac{\partial \boldsymbol{D}}{\partial t}\right) \cdot \boldsymbol{E}\mathrm{d}^3 r \\ &+ \int_{C\subset\Omega} \boldsymbol{J} \cdot \left(\boldsymbol{E} - \sigma^{-1}\boldsymbol{J} + \mathbf{V} \times \boldsymbol{B}\right) \mathrm{d}^3 r \\ &+ P_{\mathrm{Joule}}(\Omega) + \frac{\mathrm{d}F}{\mathrm{d}t}(\Omega) \\ &- \sum_i \oint_{\partial\Omega_i} ([\boldsymbol{E} \times \boldsymbol{H}] \cdot \mathbf{n})\, \mathrm{d}^2 r \\ &+ \sum_i \oint_{\partial\Omega_i} \left(\left[\int_0^{\boldsymbol{B}} \boldsymbol{H} \cdot \delta\boldsymbol{b} + \int_0^{\boldsymbol{E}} \boldsymbol{E} \cdot \delta\boldsymbol{d}\right] (\mathbf{V}_i \cdot \mathbf{n})\right) \mathrm{d}^2 r \\ &+ \int_{C\subset\Omega} \mathbf{V} \cdot (\boldsymbol{J} \times \boldsymbol{B})\, \mathrm{d}^3 r \end{aligned} \tag{14}$$

where $F(\Omega)$ is the restriction to Ω of the Helmholtz' potential:

$$\begin{aligned} F(\Omega) &= G(\Omega) + \int_{\Omega} (\boldsymbol{H} \cdot \boldsymbol{B} + \boldsymbol{E} \cdot \boldsymbol{D})\, \mathrm{d}^3 r \\ &= \int_{\Omega} \left(\int_0^{\boldsymbol{B}} \boldsymbol{H} \cdot \delta\boldsymbol{b} + \int_0^{\boldsymbol{D}} \boldsymbol{E} \cdot \delta\boldsymbol{d}\right) \mathrm{d}^3 r \end{aligned} \tag{15}$$

which matches the electromagnetic energy F of the field at thermostatistic equilibrium of the Gibbs potential (5).

At the minimum of the functional (1), the Maxwell equation set and Ohm's law (9) are checked so that:

- the first three residual terms vanish in (14). After some tedious first order calculations involving motion-induced interface discontinuities according to Eqs. (3) and (11), the last three expressions in (14) provides the mechanical power:

$$\begin{aligned} \mathrm{P}_{\mathrm{mech}}(\Omega) = &\sum_i \oint_{\partial\Omega_i} (\mathbf{n} \cdot \mathbf{V}_i)\, [\mathbf{D} \times \mathbf{B}] \cdot \mathbf{V}_i\; \mathrm{d}^2 r \\ &+ \sum_i \oint_{\partial\Omega_i} \left([(\mathbf{B} \cdot \mathbf{n})\, \mathbf{H}] - \left[\int_0^{\mathbf{H}} \mathbf{B} \cdot \delta\mathbf{h}\right] \cdot \mathbf{n}\right) \cdot \mathbf{V}_i\; \mathrm{d}^2 r \\ &+ \sum_i \oint_{\partial\Omega_i} \left([(\mathbf{D} \cdot \mathbf{n})\, \mathbf{E}] - \left[\int_0^{\mathbf{E}} \mathbf{D} \cdot \delta\mathbf{e}\right] \cdot \mathbf{n}\right) \cdot \mathbf{V}_i\; \mathrm{d}^2 r \\ &+ \sum_i \int_{C_i} (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{V}_i\; \mathrm{d}^3 r \end{aligned} \tag{16}$$

where the first term denotes a vanishing "impulsion" term within the quasi-static approximation; the second and the third one gathers the switching permeance effects occurring at the various interfaces $\partial\Omega_i$ of the domain Ω, and the fourth one is related to the power of the Laplace's force. For conductors with linear magnetic behavior law, the last three contributions may be gathered in the Maxwell's stress tensor. As a result, the relation (14) matches the integral form of the Poynting's conservation equation:

$$\mathrm{P_{elec}}\left(\Omega\right) = \mathrm{P_{Joule}}\left(\Omega\right) + \frac{\mathrm{dF}}{\mathrm{d}t}\left(\Omega\right) + \mathrm{P_{mech}}\left(\Omega\right) \tag{17}$$

Hence,

- the contribution of Ω to (1) reads:

$$-\mathrm{P_{mech}}\left(\Omega\right) - \frac{\mathrm{dG}}{\mathrm{d}t}\left(\Omega\right) + \mathrm{P_{elec}}\left(\Omega\right) \tag{18}$$

The previous approach addresses a thermodynamic-oriented justification of the Finite Element Method, which consists in building an approximation of the stationary conditions expressed at Eqs. (1) and (5) but with a finite number of degrees of freedom.

III. Quasi-static regimes

Besides, the set of transformation laws (7), (8), (6) and (12) is not compatible with the invariance of the behavior laws, at least in vacuum. As quasi-static phenomena are observed, it is mandatory to admit that one coupling term in (1) is much more prevalent than the other according to the studied domain. As a result, the quasi-static limit must be split in magnetic and electric regimes (Table I) [1]. It should be noticed that conductors are always depicted within the magnetic regime of the quasi-static limit.

The signal/power integrity issue in Electro-Magnetic Compatibility may be viewed as a transition in (1) between the magnetic- and the electrostatic-driven coupling in a given domain of the dielectric region, the actual regime corresponding to the most reversible evolution of the global system, *i.e.* consolidated from the contributions (18) to the functional (1) in Table I.

As a general rule, phase transition is associated with a break in the symmetry of the problem [9]. This point is discussed from the magnitude of the various times occurring in any electrodynamic problem of size ℓ [10][11], namely the magnetic diffusion time $\tau_{\mathrm{m}} = \sigma\mu\ell^2$ and the charge life duration $\tau_{\mathrm{e}} = \frac{\varepsilon}{\mu}$ occurring in conductor; and the transit time of an electromagnetic wave crossing the system $\tau_{\mathrm{em}} = \sqrt{\sigma\mu}\ell$ checking the relation $\tau_{\mathrm{em}}^2 = \tau_{\mathrm{e}}\tau_{\mathrm{m}}$. These values could be faced to the typical duration τ of the excitation so that the various regimes may be schematically stressed (Fig. 1) [12].

Defining the flux density and the electric field from the potentials $(\mathbf{A}, \mathrm{V})$ according to:

$$\mathbf{B} = \mathbf{curl}\,\mathbf{A} \tag{19}$$

$$\mathbf{E} = -\mathbf{grad}\mathrm{V} - \frac{\partial\mathbf{A}}{\partial t} \tag{20}$$

various gauges may be considered according to the quasi-static regime to decouple the evolution equations expressed on the potentials. As a result, the transition between the QS-magnetic and the QS-electric regimes corresponds respectively to a change from the Stratton's to the Lorentz' gauges driven by the electrical angular velocity $\varpi = \frac{2\pi}{\tau}$ [13].

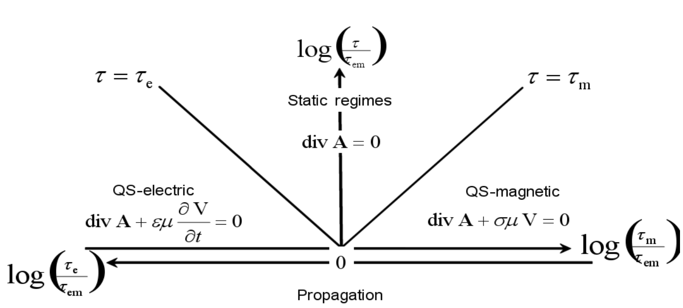

Fig. 1. Domains of validity of the Quasi-static regimes with their gauges conditions. Notice the absence of border between the electric and magnetic regimes which reinforces the existence of a transition driven by the gauges.

IV. Multi-scale issue

Homogenization techniques consist in replacing an original problem described from deep within material by an equivalent one where the rapidly space-varying lengths at the design scale are discarded. From that viewpoint, macroscopic electromagnetism is already an homogeneized theory, but further homogeneizations could be carried out to take advantage of the spatial regularity occuring in various media. The benefits of homogenization are obvious: While taking into account the fine structure would be a huge or impossible task, the equivalent problem may be solved more cheaply, with a reasonable mesh size and a better conditioning. However a challenging task of any homogeneization procedure consists in deriving unconventional behavior laws from a basic resolution at a "unit cell" scale in order to restore properly the dissipative phenomena occuring therein.

As matter of fact, the basis of the free current density $\mathbf{j}$ in regular conducting structures involves a superposition of compact distributions of microscopic currents $\mathbf{j}_w$ located within each "unit cell" Ω_w at position $\mathbf{r}_w$, with a density $n_w(\mathbf{r})$; and a possible macroscopic free current density $\mathbf{J}$. This description leads to a spectral decomposition splitted in well-defined peaks from the microscopic to the network scales and allows a *mean-field* hypothesis [9] which consists in performing embedded minimizations for $k \gtrless k_{lp}$. Whereas the low-pass filtering process $(k < k_{lp})$ is equivalent to a smoothing procedure on the various fields [14], the high frequencies restore the deepest magnetodynamic mechanisms. Hence, the restriction to C of the minimization in the RHS of (1) becomes:

$$\min_{\langle \boldsymbol{h}\rangle(t)} \int_C \Big(\sigma^{-1}\left(\mathbf{curl}\,\langle\boldsymbol{h}\rangle\right)^2 + \partial_t\left(\langle\mathbf{b}\rangle\cdot\langle\boldsymbol{h}\rangle\right)\Big)\,\mathrm{d}^3r \tag{21a}$$

$$+ \sum_{w\in C} \min_{\boldsymbol{h}_\mu(t)} \int_{\Omega_w} \Big(\sigma^{-1}\left(\mathbf{curl}\boldsymbol{h}_\mu\right)^2 + \partial_t\left(\mathbf{b}_\mu\cdot\boldsymbol{h}_\mu\right)\Big)\,\mathrm{d}^3r \tag{21b}$$

where $\langle\boldsymbol{h}\rangle$ (resp. $\langle\mathbf{b}\rangle$) denotes the averaged magnetic field (resp. averaged flux density) and is obtained through a convolution by the low-pass filter ρ_{lp}:

$$\langle\boldsymbol{h}\rangle(\mathbf{r}) = \int \boldsymbol{h}\left(\mathbf{x}'\right)\rho_{lp}\left(\mathbf{r}-\mathbf{r}'\right)\mathrm{d}^3r' \tag{22}$$

TABLE I

REGIMES ALLOWED WITHIN THE QUASI-STATIC LIMIT (FROM [1]). THE RESOLUTION IS PERFORMED IN TWO STEPS: THE FIRST (SECOND) ONE INVOLVES THE DOMINANT (MARGINAL) COUPLING. THE ARROW ($\Longleftarrow$) DENOTES A FIELD OBTAINED FROM THE MAIN RESOLUTION WHICH ACTS AS A SOURCE FOR THE MARGINAL ONE.

	Magnetic regime $\lvert\mathbf{B}\cdot\mathbf{H}\rvert \gg \lvert\mathbf{D}\cdot\mathbf{E}\rvert$	Electric regime $\lvert\mathbf{B}\cdot\mathbf{H}\rvert \ll \lvert\mathbf{D}\cdot\mathbf{E}\rvert$
Conductors	$\mathbf{curl}\left(\sigma^{-1}\mathbf{curl}\,\mathbf{H}\right) = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}$ $\mathrm{div}\mathbf{B} = 0$ $\mathbf{curl}\,\mathbf{H} = \mathbf{J}$ $\mathbf{B} = \mu(\mathbf{H})\cdot\mathbf{H}$	N/A
Dielectrics	$\mathrm{div}\mathbf{B} = 0$ $\mathbf{curl}\,\mathbf{H} = 0$ $\mathbf{B} = \mu(\mathbf{H})\cdot\mathbf{H}$	$\mathrm{div}\mathbf{D} = 0$ $\mathbf{curl}\,\mathbf{E} = 0$ $\mathbf{D} = \varepsilon(\mathbf{E})\cdot\mathbf{E}$
	$\mathbf{curl}\,\mathbf{E} \Leftarrow -\frac{\partial\mathbf{B}}{\partial t}$ $\mathrm{div}\mathbf{D} = 0$ $\mathbf{D} = \varepsilon(\mathbf{E})\cdot\mathbf{E}$ on $\partial C:\quad \mathbf{n}\times\mathbf{E} \Leftarrow \mathbf{n}\times\left(\sigma^{-1}\mathbf{J} - \mathbf{V}\times\mathbf{B}\right)$	$\mathbf{curl}\,\mathbf{H} \Leftarrow \frac{\partial\mathbf{D}}{\partial t}$ $\mathrm{div}\mathbf{B} = 0$ $\mathbf{B} = \mu(\mathbf{H})\cdot\mathbf{H}$ on $\partial C:\quad \mathbf{n}\times[\mathbf{H}] \Leftarrow \mathbf{n}\times(\mathbf{V}\times\mathbf{D})$
Contribution to the funtional (1) according to (18)	$-\mathrm{P_{mech}}(\Omega) + \mathrm{P_{elec}}(\Omega) - \frac{\mathrm{dG_m}(\Omega)}{\mathrm{d}t}$	$-\mathrm{P_{mech}}(\Omega) + \mathrm{P_{elec}}(\Omega) - \frac{\mathrm{dG_e}(\Omega)}{\mathrm{d}t}$

Similarly, $\boldsymbol{h}_\mu$ (resp. $\mathbf{b}_\mu$) stands for the rapidly space-varying part of the magnetic field (resp. flux density) and is obtained thanks to a convolution by a high-pass filter. Hence, the consistency between low- and high-spatial frequencies is enforced by:

- the conservation of the magnetic flux density after the smoothing procedure, *i.e.* $\mathbf{B} = \langle\mathbf{b}\rangle$;
- the modification of the Maxwell-Ampere equation according to the dynamic behavior of independent unit cells given by the minimization (21b). After the smoothing procedure on the microscopic eddy currents at the first order, the averaged magnetic field $\langle\boldsymbol{h}\rangle$ is shifted by the microscopic dynamic eddy currents to check a modified Maxwell-Ampere equation:

$$\mathbf{curl}\left(\langle\boldsymbol{h}\rangle + \sigma\Lambda^2\partial_t\mathbf{B}\right) = \boldsymbol{J} \tag{23}$$

where Λ denotes a length parameter lumping *locally* all the complexity due to spatial dependence of $\boldsymbol{j}_w$ [15].

Hence, the Parseval's theorem [16] provides:

$$\sum_{w\in C}\min_{\boldsymbol{h}_\mu(t)}\int_{\Omega_w}\left(\sigma^{-1}\left(\mathbf{curl}\,\boldsymbol{h}_\mu\right)^2 + \partial_t\left(\mathbf{b}_\mu\cdot\boldsymbol{h}_\mu\right)\right)\mathrm{d}^3r = \int_C \mathbf{B}\cdot\partial_t\left(\boldsymbol{H} - \langle\boldsymbol{h}\rangle\right)\mathrm{d}^3r \tag{24}$$

so that the functional (1) is robust at any scale provided that the expression of the losses (4) is updated after each homogenization procedure.

V. CONCLUSIONS

The loss of power/signal integrity is explained as a transition between the QS-magnetic and QS-electric regimes. Moreover, thermodynamics provides the contributions (18) to the functional (1) from which the discussion should be achieved (Table I). Besides, the formulation derived from thermodynamics is robust at any scale, mainly due to the quadratic property of the functionals (1) and (5) and the well-splitted spectral decomposition of the current density. Such an approach should be valuable to reconcile low and high frequency studies within the power management framework.

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