Electromagnetic Media with no Dispersion Equation

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***Abstract*—Recently, some novel boundary conditions have been observed to arise at interfaces of certain electromagnetic media in which plane waves are not restricted by a dispersion equation. In the present study an attempt is made to define most general media in which dispersion equations are identically satisfied for any plane wave. Applying four-dimensional formalism, it is shown that there are three classes of media satisfying this requirement.**

I. Introduction

Time-harmonic electromagnetic plane waves in linear homogeneous media are characterized by exponential spatial dependence as

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}\exp(-j\boldsymbol{k}\cdot\boldsymbol{r}), \quad (1)$$
$$\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{H}\exp(-j\boldsymbol{k}\cdot\boldsymbol{r}), \quad (2)$$

where the wave vector $\boldsymbol{k}$ is restricted by a dispersion equation

$$D(\boldsymbol{k}) = 0, \quad (3)$$

depending on the parameters of the medium and the angular frequency ω. In the general case, the dispersion equation is of the fourth order. For some special media the fourth-order equation is reduced to two second order equations,

$$D(\boldsymbol{k}) = D_1(\boldsymbol{k})D_2(\boldsymbol{k}), \quad \Rightarrow \quad D_1(\boldsymbol{k}) = 0, \;\; D_2(\boldsymbol{k}) = 0. \quad (4)$$

The uniaxially anisotropic medium may serve as an example, because any wave can be decomposed in TE and TM partial waves each of which obeys a dispersion equation of the second order. In some more special media the two second-order equations are the same,

$$D(\boldsymbol{k}) = D_1(\boldsymbol{k})D_1(\boldsymbol{k}), \quad \Rightarrow \quad D_1(\boldsymbol{k}) = 0, \quad (5)$$

in which case the medium is nonbirefringent. An obvious example of such a medium is the isotropic medium. Finally, there may exist media for which the dispersion equation is identically satisfied for any wave vector $\boldsymbol{k}$. This means that, for a plane wave in such a medium, $\boldsymbol{k}$ can be arbitrarily chosen. As a simple example, $D(\boldsymbol{k}) = \boldsymbol{k}\cdot\overline{\overline{\mathsf{A}}}\cdot\boldsymbol{k} = 0$ is identically satisfied for any $\boldsymbol{k}$ when $\overline{\overline{\mathsf{A}}}$ is an antisymmetric dyadic. We could say that for a medium with such a property, there is no dispersion equation. This does not, however, mean that the medium is nondispersive. On the contrary, such a medium must be dispersive since, otherwise, for a proper choice of the $\boldsymbol{k}$ vector the group velocity in the medium would exceed the velocity of light.

Media with no dispersion equation have recently emerged when studying boundary conditions at interfaces of certain media. For example, defining a medium by conditions of the form

$$\boldsymbol{D} = (\overline{\overline{\alpha}} + \alpha\overline{\overline{\mathsf{I}}})\cdot\boldsymbol{B} + \boldsymbol{c}\times\boldsymbol{E} \quad (6)$$
$$\boldsymbol{H} = \boldsymbol{g}\times\boldsymbol{B} + (\overline{\overline{\alpha}}^T - \alpha\overline{\overline{\mathsf{I}}})\cdot\boldsymbol{E} \quad (7)$$

where $\overline{\overline{\alpha}}$ is a dyadic, $\boldsymbol{c}$ and $\boldsymbol{g}$ are two vectors and α is a scalar, the following equation is obtained for the plane wave field $\boldsymbol{E}$ through elimination of the other fields from the Maxwell equations,

$$\boldsymbol{q}(\boldsymbol{k})\times\boldsymbol{E} = 0, \quad (8)$$

with

$$\boldsymbol{q}(\boldsymbol{k}) = (\boldsymbol{g}\cdot\boldsymbol{k} - \omega\mathrm{tr}\overline{\overline{\alpha}})\boldsymbol{k} + \omega\boldsymbol{k}\cdot\overline{\overline{\alpha}} + \omega^2\boldsymbol{c}. \quad (9)$$

In more general media, choice of $\boldsymbol{k}$ would lead to vanishing of $\boldsymbol{E}$. However, in (8) $\boldsymbol{k}$ may be freely chosen, after which the polarization of $\boldsymbol{E}$ becomes parallel to the vector $\boldsymbol{q}(\boldsymbol{k})$ and the rest of the field vectors are obtained from the Maxwell equations. In [1] it was shown that a uniaxial version of such a medium yields DB boundary conditions at its interface, while for another choice of parameters a generalization of soft-and-hard and DB conditions can be obtained at the interface [2].

It is the purpose of this paper to study which media have the property of being free from a dispersion equation. It appears convenient to do the analysis applying the four-dimensional formalism of reference [3].

II. Dispersion equation

The Maxwell equations outside sources can be expressed in a compact form as

$$\boldsymbol{d}\wedge\boldsymbol{\Phi} = 0, \quad (10)$$
$$\boldsymbol{d}\wedge\boldsymbol{\Psi} = 0, \quad (11)$$

where the 4D electromagnetic two-forms $\boldsymbol{\Phi}, \boldsymbol{\Psi} \in \mathbb{F}_2$ are defined in terms of 3D (spatial) one-forms $\boldsymbol{E}, \boldsymbol{H}$ and two-forms $\boldsymbol{B}, \boldsymbol{D}$ as

$$\boldsymbol{\Phi} = \boldsymbol{B} + \boldsymbol{E}\wedge\boldsymbol{\varepsilon}_4, \quad (12)$$
$$\boldsymbol{\Psi} = \boldsymbol{D} - \boldsymbol{H}\wedge\boldsymbol{\varepsilon}_4. \quad (13)$$

For details in the notation, see [3]. A plane wave has an exponential dependence on the space-time vector $\boldsymbol{x}$,

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{\Phi}\exp(\boldsymbol{\nu}|\boldsymbol{x}), \tag{14}$$

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \boldsymbol{\Psi}\exp(\boldsymbol{\nu}|\boldsymbol{x}), \tag{15}$$

where $\boldsymbol{\nu} \in \mathbb{F}_1$ is the wave one-form whose spatial part corresponds to the $\boldsymbol{k}$ vector above. For a plane wave the Maxwell equations (10), (11) become

$$\boldsymbol{\nu}\wedge\boldsymbol{\Phi} = 0, \tag{16}$$

$$\boldsymbol{\nu}\wedge\boldsymbol{\Psi} = 0. \tag{17}$$

Hence, we can express the field two-forms in terms of potential one-forms $\boldsymbol{\phi}, \boldsymbol{\psi}$ as

$$\boldsymbol{\Phi} = \boldsymbol{\nu}\wedge\boldsymbol{\phi}, \qquad \boldsymbol{\Psi} = \boldsymbol{\nu}\wedge\boldsymbol{\psi}. \tag{18}$$

A linear medium can be represented as a linear mapping between the electromagnetic one-forms in terms of a medium bidyadic $\overline{\overline{\mathsf{M}}} \in \mathbb{F}_2\mathbb{E}_2$ as

$$\boldsymbol{\Psi} = \overline{\overline{\mathsf{M}}}|\boldsymbol{\Phi}, \tag{19}$$

or in terms of a modified medium bidyadic $\overline{\overline{\mathsf{M}}}_m \in \mathbb{E}_2\mathbb{E}_2$ as

$$\boldsymbol{e}_N\lfloor\boldsymbol{\Psi} = \overline{\overline{\mathsf{M}}}_m|\boldsymbol{\Phi}. \tag{20}$$

The two bidyadics have the relation $\overline{\overline{\mathsf{M}}}_m = \boldsymbol{e}_N\lfloor\overline{\overline{\mathsf{M}}}$.

From (17) the condition for the potential one-form $\boldsymbol{\phi}$ becomes

$$\boldsymbol{\nu}\wedge\boldsymbol{\Psi} = (\boldsymbol{\nu}\wedge\overline{\overline{\mathsf{M}}}\lfloor\boldsymbol{\nu})|\boldsymbol{\phi} = 0, \tag{21}$$

where the dyadic in brackets belongs to the space $\mathbb{F}_3\mathbb{E}_1$. It defines the dispersion dyadic $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) \in \mathbb{E}_1\mathbb{E}_1$ as

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) = \boldsymbol{e}_N\lfloor(\boldsymbol{\nu}\wedge\overline{\overline{\mathsf{M}}}\lfloor\boldsymbol{\nu}) = -\boldsymbol{\nu}\rfloor(\boldsymbol{e}_N\lfloor\overline{\overline{\mathsf{M}}})\lfloor\boldsymbol{\nu} = \overline{\overline{\mathsf{M}}}_m\lfloor\lfloor\boldsymbol{\nu}\boldsymbol{\nu}. \tag{22}$$

Because of (21) and (22), the dispersion dyadic satisfies

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})|\boldsymbol{\phi} = 0, \tag{23}$$

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})|\boldsymbol{\nu} = 0. \tag{24}$$

For $\boldsymbol{\Phi} = \boldsymbol{\nu}\wedge\boldsymbol{\phi} \neq 0$ the one-forms $\boldsymbol{\phi}$ and $\boldsymbol{\nu}$ are linearly independent, whence the rank of the dispersion dyadic must be less than three. Thus, the dispersion dyadic must satisfy

$$\begin{aligned}\overline{\overline{\mathsf{D}}}^{(3)}(\boldsymbol{\nu}) &= \frac{1}{6}\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})^\wedge_\wedge\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})^\wedge_\wedge\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) \\ &= (\boldsymbol{e}_N\boldsymbol{e}_N\lfloor\lfloor\boldsymbol{\nu}\boldsymbol{\nu})D(\boldsymbol{\nu}) = 0.\end{aligned} \tag{25}$$

Actually, (25) is equivalent to a scalar dispersion equation (known also as Fresnel equation). It can be given the following explicit form [4],

$$D(\boldsymbol{\nu}) = \frac{1}{6}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N||(\overline{\overline{\mathsf{M}}}_m{}^\wedge_\wedge(\boldsymbol{\nu}\boldsymbol{\nu}\rfloor\rfloor(\overline{\overline{\mathsf{M}}}_m{}^\wedge_\wedge(\boldsymbol{\nu}\boldsymbol{\nu}\rfloor\rfloor\overline{\overline{\mathsf{M}}}_m)))) = 0, \tag{26}$$

which is of the fourth order in $\boldsymbol{\nu}$.

III. Media with no dispersion equation

A medium does not have a dispersion equation if the dispersion dyadic $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})$ satisfies (25) for all possible one-forms $\boldsymbol{\nu}$, i.e., if it is at most of rank 2.

To study various possibilities, we first assume $\overline{\overline{\mathsf{D}}}^{(2)}(\boldsymbol{\nu}) \neq 0$ for all $\boldsymbol{\nu}$, whence the rank of $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})$ is exactly 2. Such a dyadic can be expanded in two dyadic terms as

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) = \overline{\overline{\mathsf{M}}}_m\lfloor\lfloor\boldsymbol{\nu}\boldsymbol{\nu} = \boldsymbol{a}\boldsymbol{c} + \boldsymbol{b}\boldsymbol{d}, \tag{27}$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are four vector functions of $\boldsymbol{\nu}$ satisfying $\boldsymbol{a}\wedge\boldsymbol{b} \neq 0$ and $\boldsymbol{c}\wedge\boldsymbol{d} \neq 0$. Since $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})$ is quadratic in $\boldsymbol{\nu}$, there are a few possibilities. If all four vectors are linear functions of $\boldsymbol{\nu}$ and because they satisfy $\boldsymbol{a}|\boldsymbol{\nu} = 0$ etc., they can be expressed as

$$\boldsymbol{a} = \boldsymbol{A}\lfloor\boldsymbol{\nu}, \quad \boldsymbol{b} = \boldsymbol{B}\lfloor\boldsymbol{\nu}, \quad \boldsymbol{c} = \boldsymbol{C}\lfloor\boldsymbol{\nu}, \quad \boldsymbol{d} = \boldsymbol{D}\lfloor\boldsymbol{\nu}, \tag{28}$$

in terms of four bivectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$. This yields the first possible class of media with no dispersion equation:

$$\overline{\overline{\mathsf{M}}}_m = \boldsymbol{A}\boldsymbol{C} + \boldsymbol{B}\boldsymbol{D} + \alpha\boldsymbol{e}_N\lfloor\overline{\overline{\mathsf{I}}}^{(2)T}, \tag{29}$$

or

$$\overline{\overline{\mathsf{M}}} = \boldsymbol{\Pi}\boldsymbol{C} + \boldsymbol{\Delta}\boldsymbol{D} + \alpha\overline{\overline{\mathsf{I}}}^{(2)T}, \tag{30}$$

where $\boldsymbol{\Pi}, \boldsymbol{\Delta}$ are some two-forms. The last term in either expression does not affect the dispersion equation for any α.

As a second possibility we may assume that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in (27) are quadratic functions of $\boldsymbol{\nu}$ while $\boldsymbol{c}$ and $\boldsymbol{d}$ are constant. It turns out that in this case the dispersion dyadic must be of the form

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) = \boldsymbol{e}_N\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\overline{\overline{\mathsf{P}}}^T), \tag{31}$$

where the one-forms $\boldsymbol{\alpha}, \boldsymbol{\beta}$ depend on $\boldsymbol{\nu}$ while $\overline{\overline{\mathsf{P}}} \in \mathbb{E}_1\mathbb{F}_1$ may be any dyadic independent of $\boldsymbol{\nu}$ and satisfying $\overline{\overline{\mathsf{P}}}^{(2)} \neq 0$. From the conditions $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})|\boldsymbol{\nu} = 0$ and $\boldsymbol{\nu}|\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) = 0$ the dependence of one-forms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ on $\boldsymbol{\nu}$ can be figured out. There are actually two cases, either

$$\overline{\overline{\mathsf{M}}} = M\overline{\overline{\mathsf{P}}}^{(2)T} + \alpha\overline{\overline{\mathsf{I}}}^{(2)T}, \tag{32}$$

or

$$\overline{\overline{\mathsf{M}}} = (\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{I}}})^T, \tag{33}$$

defined by dyadics $\overline{\overline{\mathsf{P}}}, \overline{\overline{\mathsf{B}}} \in \mathbb{E}_1\mathbb{F}_1$. For $\alpha = 0$, the solution (32) corresponds to a set of media studied peviously under the name P-media [5], while the latter solution (33) corresponds to the class of skewon-axion media [6] (or IB-media [7]), whose medium conditions can be expressed in 3D form as (6), (7).

Because the other possibilities of (27) seem to lead either one of the above cases or nowhere, it is concluded that, when the dispersion dyadic is of rank 2 for all possible one-forms $\boldsymbol{\nu}$, expressions of the form (30), (32) and (33) define three possible classes of media with no dispersion equation.

It remains to study the possibility when the dispersion dyadic is of rank 1, i.e., it satisfies $\overline{\overline{\mathsf{D}}}^{(2)}(\boldsymbol{\nu}) = 0$, in which case it can be represented by a single dyadic term as

$$\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu}) = \overline{\overline{\mathsf{M}}}_m\lfloor\lfloor\boldsymbol{\nu}\boldsymbol{\nu} = \boldsymbol{a}\boldsymbol{c}. \tag{34}$$

Assuming $\boldsymbol{a}$ and $\boldsymbol{c}$ linear functions of $\boldsymbol{\nu}$ we arrive at a special case of (30) with $\boldsymbol{\Delta D} = 0$. Assuming that $\boldsymbol{a}$ is a quadratic function of $\boldsymbol{\nu}$ leads to special cases of (32) and (33) which reduce to the bare axion medium, $\overline{\overline{\mathsf{M}}} = \alpha \bar{\bar{\mathsf{I}}}^{(2)T}$.

IV. Conclusion

As a conclusion, we have studied the possibilities of defining classes of media for which the dispersion equation $D(\boldsymbol{\nu}) = 0$ normally restricting the choice of the wave one-form $\boldsymbol{\nu}$ of a plane wave, is satisfied identically for any $\boldsymbol{\nu}$. Since the rank of the dispersion dyadic $\overline{\overline{\mathsf{D}}}(\boldsymbol{\nu})$ must be less than three, we have considered the cases when it it either two or one for all $\boldsymbol{\nu}$. The outcome was that there are three classes of possible media, defined by the form of their medium bidyadics, in four-dimensional formalism of [3], as (30), (32) and (33).

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