Electromagnetic Energies in Time Domain – the Dipole Case

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***Abstract*— Closed form expressions can be derived in time domain to calculate the electric and magnetic field energy stored around a device and the energy radiated by the device. In this paper these expressions are given, without proof, and they are illustrated through numerical calculations for the most basic radiator type: the dipole.**

I. Introduction

The electric and magnetic energies stored around a device have been studied mainly in the frequency domain. Although the first papers were published about 50-60 years ago [1], [2], and many authors have been investigating Q factors since then [3], [4], [5], [6], [7], even in frequency domain it was only recently that general expressions were formulated to calculate newly defined reactive energies explicitly in terms of the currents [8]. These expressions have led to new, more effective studies of Q [9], [10], [11].

The situation is quite different in time domain. To the knowledge of the authors, the only paper that thoroughly studies electric and magnetic energies in time domain is [13]. Just as many authors do in frequency domain, Shlivinski uses a decomposition of the field into spherical modes in time domain. This automatically means that the resulting analysis is an approximation, taking into account only the energy outside a spherical surface with radius *a*. Typically, this radius is the smallest radius that completely circumscribes the radiator. It can be shown that general and rigorous expressions can be derived that do not rely on this decomposition.

II. Energy Expressions

Consider an arbitrary current distribution $\mathbf{J}_s$ flowing in a volume V_s and time interval T_s embedded in free space with permittivity ε_0 and permeability μ_0. The expressions for the electric and magnetic energy distributed over space are (the proof is beyond the scope of this paper)

$$W^e = W^e_\rho + \frac{W_{rad}}{2} + W_{osc} \qquad (1)$$

$$W^m = W^m_J + \frac{W_{rad}}{2} - W_{osc} \qquad (2)$$

with

$$W^e_\rho = \frac{1}{8\pi\varepsilon_0}\int_{V_1}\int_{V_2} \frac{\rho_1(t-\frac{r_{21}}{c})\rho_2(t)}{r_{21}} dV_2 dV_1 \qquad (3)$$

$$W^m_J = \frac{1}{8\pi\varepsilon_0}\frac{1}{c^2}\int_{V_1}\int_{V_2} \frac{\mathbf{J}_1(t-\frac{r_{21}}{c})\cdot\mathbf{J}_2(t)}{r_{21}} dV_2 dV_1 \qquad (4)$$

$$W_{rad} = \frac{1}{8\pi\varepsilon_0}\int_{V_1}\int_{V_2} \frac{I^{rad}}{r_{21}} dV_2 dV_1 \qquad (5)$$

$$W_{osc} = \frac{1}{16\pi\varepsilon_0}\int_{V_1}\int_{V_2} \frac{I^{osc}}{r_{21}} dV_2 dV_1 \qquad (6)$$

$$\begin{aligned} I^{rad} = & \int_{\frac{r_{21}}{2c}}^{t-\frac{r_{21}}{2c}} \left(\rho_1(t_1-\frac{r_{21}}{2c})\frac{\partial\rho_2(t_1+\frac{r_{21}}{2c})}{\partial t_1} - \rho_1(t_1+\frac{r_{21}}{2c})\frac{\partial\rho_2(t_1-\frac{r_{21}}{2c})}{\partial t_1}\right)dt_1 \\ & -\frac{1}{c^2}\int_{\frac{r_{21}}{2c}}^{t-\frac{r_{21}}{2c}} \left(\mathbf{J}_1(t_1-\frac{r_{21}}{2c})\cdot\frac{\partial\mathbf{J}_2(t_1+\frac{r_{21}}{2c})}{\partial t_1} - \mathbf{J}_1(t_1+\frac{r_{21}}{2c})\cdot\frac{\partial\mathbf{J}_2(t_1-\frac{r_{21}}{2c})}{\partial t_1}\right)dt_1 \end{aligned} \qquad (7)$$

$$\begin{aligned} I^{osc} = & \int_{-\frac{r_{21}}{2c}}^{\frac{r_{21}}{2c}} \rho_1(t-t_1-\frac{r_{21}}{2c})\frac{\partial\rho_2(t+t_1-\frac{r_{21}}{2c})}{\partial t_1}dt_1 \\ & +\frac{1}{c^2}\int_{-\frac{r_{21}}{2c}}^{\frac{r_{21}}{2c}} \left(\mathbf{J}_1(t-t_1-\frac{r_{21}}{2c})\cdot\frac{\partial\mathbf{J}_2(t+t_1-\frac{r_{21}}{2c})}{\partial t_1}\right)dt_1 \end{aligned} \qquad (8)$$

The subscripts 1 and 2 both replace the subscript *s*. They are used to indicate a double integration over the volume. This means thus that $V_1 = V_2 = V_s$. r_{21} is the distance between points 1 and 2 and $c = 1/\sqrt{\mu_0 \varepsilon_0}$ is the speed of light in free space. The charge distribution generated by the current is $\rho_s = -\int_{t_s} \nabla \cdot \mathbf{J}_s \, dt_s$.

It is seen that the energies can be considered as the sum of several terms. The first terms in (1) and (2) are not related to radiation. They are related to the energy linked directly to the charge and current distribution in space. The second term in (1) and (2) is related to radiation. The third term represents an oscillation between electric and magnetic energy.

III. Numerical Examples

Consider a dipole of length L = 1 m and radius a = 0.5 mm excited by a sinusoidal current at frequency *freq*, starting at *t=0*. Along the dipole length, the current has the current profile

$$I(z) = I_0 \sin(k(\frac{L}{2} - |z|)) \quad \text{for} \quad |z| \le \frac{L}{2} \qquad (9)$$
$$= 0 \quad \text{otherwize}$$

with $k = \omega\sqrt{\varepsilon_0 \mu_0}$. It is easily checked that a pure sinusoidal current starting at $t = 0$ would generate a charge that shows a DC component on top of the sinusoidal component. Obviously, this cannot be maintained in a sinusoidal steady state. Physically, the DC charge will leak away, yielding the pure sinusoidal behavior. The current flowing in the system is therefore modeled by the current flowing in a parallel RC circuit. The C is a simplified model for the dipole, and the R is the parallel path over which the DC charge is leaking away. The solution for the current shows the time dependency

$$f(t) = \sin(\omega t) - \frac{\sin(\omega t) - \omega RC(\cos(\omega t) - e^{-\frac{t}{RC}})}{1 + (\omega RC)^2} \quad \text{for} \quad t \ge 0 \qquad (10)$$
$$= 0 \quad \text{otherwize}$$

Clearly a sinusoidal steady state can be seen. It is a combination of a sine and a cosine. The transient can also be seen. Note that this model to derive the current is clearly an approximation, but it is sufficient to illustrate the important phenomena occurring. In order to clearly show what is happening, the RC is chosen in such a way that the exponential function is reduced to *1/e* of its initial value after two periods of the sine, which means that $\omega RC = 4\pi$.

The energies as a function of time, numerically calculated with (3) – (6), for several frequencies are depicted in Fig. 1. Also depicted is the average power radiated over the time frame considered

$$P_{rad}^{av} = \frac{W_{rad}}{t} \qquad (11)$$

Time is given relative to T, which equals *L/c* for the dipole. Several fundamental properties are clearly numerically illustrated. First of all, W_ρ^e, W_j^m, and W_{rad} (and thus also P_{rad}^{av}), numerically seem to be always positive. W_{osc} is the only energy which systematically oscillates between positive and negative values.

Both W_ρ^e and W_j^m show a transient behavior. It is seen that during this transient, their maxima alternate between values higher and values lower than their steady state maxima. This is much more pronounced for W_ρ^e. Remark the huge buildup of W_ρ^e in the beginning. This is due to the fact that the DC charge did not have a chance to leak away yet.

Note that for 150 MHz and 450 MHz, the amplitudes of W_ρ^e and W_j^m tend to the same steady state value. This is well-known from frequency domain. It corresponds to the fact that the antenna impedance becomes purely real. Also W_{rad} is affected by the transient. It shows oscillations decreasing with time, finally also reaching a steady state linear rise corresponding to the radiated power given in frequency domain. W_{rad} and to a lesser extent also W_{osc} are depending on frequency. Note that for the dipole W_{rad} in the very beginning is increasing more slowly than in steady state.

Fig. 1 (bottom right) also shows results for a finite pulse, one sinusoidal period long. It is seen that, as soon as the pulse stops, W_ρ^e and W_j^m become zero and W_{rad} becomes constant. W_{osc} dies out a little bit later.

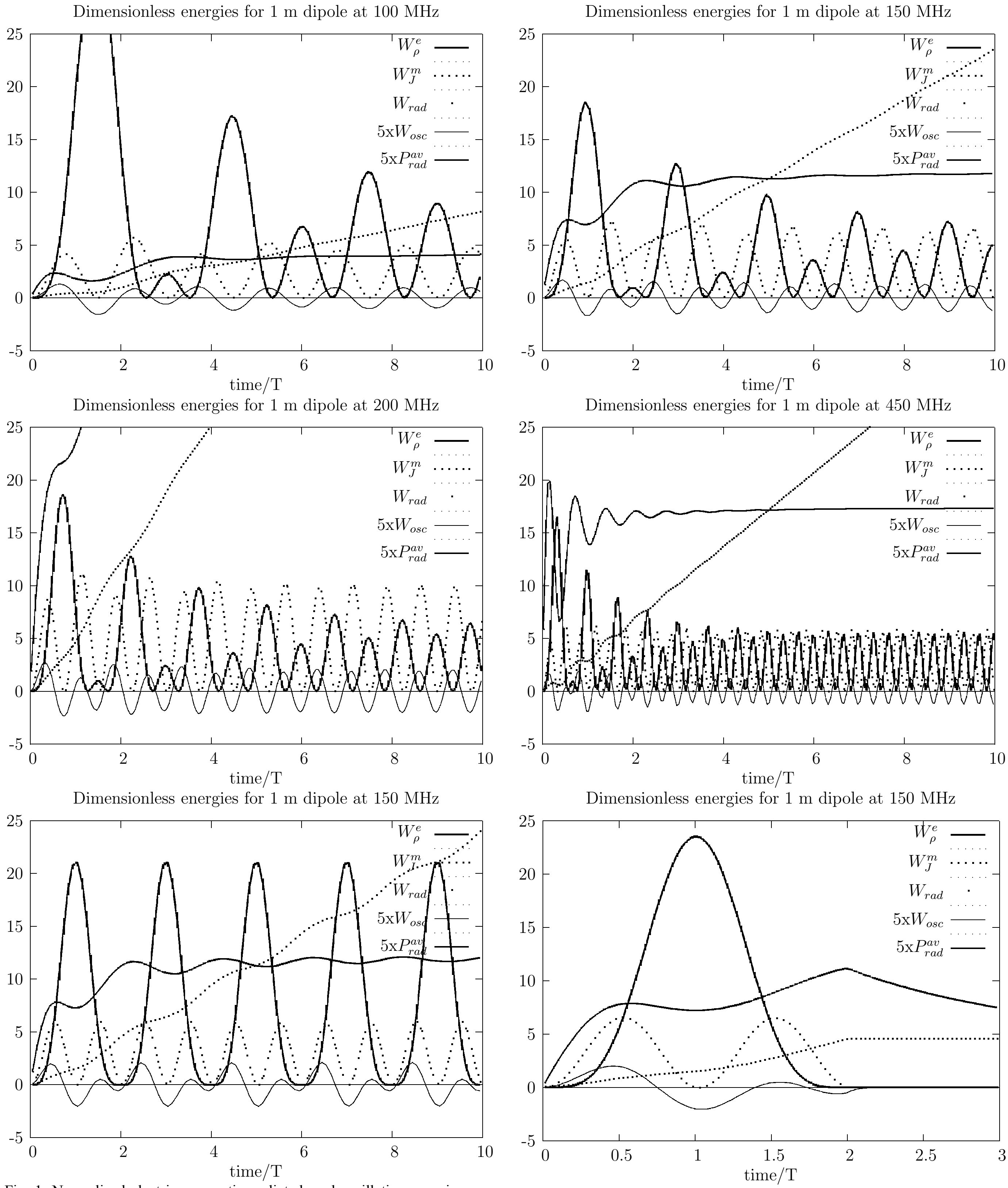

Fig. 1. Normalized electric, magnetic, radiated, and oscillating energies as a function of normalized time ($T = L/c$). The normalization is performed by multiplying with $(16\pi\omega^2\varepsilon_0)/(kI_0^2)$, which makes the energies dimensionless. The top 4 figures are for a sinusoidal excitation starting at *t=0*, thus showing a transient, since the DC component is "leaking" away through a resistor *R*. The bottom left figure is for $R = \infty$. The bottom right figure is for a sinusoidal pulse of duration 1 period.

As illustration of the effect of the DC charge, Fig. 1 (bottom left) also shows the energies in case that the DC charge does not leak away. It is seen that no real sinusoidal steady state is reached. The radiated power keeps oscillating and the behavior of W_{osc} shows a superposition of two oscillations.

IV. Conclusion

Rigorous expressions for the stored electric, stored magnetic, and radiated energies generated by an arbitrary source or device in time domain, are applied to a basic radiator: the dipole, including its starting-up phase. This study has brought a fundamental understanding of how these energies behave as a function of time.

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