Buried Object Detection by Means of a L^p Banach-Space Inversion Procedure

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Abstract—Electromagnetic imaging of buried targets is an important task that arises in several applicative fields, such as civil engineering and archeology. In the present paper, an algorithm based on a regularizing approach in L^p Banach spaces is applied to the integral equations of the inverse scattering problem. The effectiveness of the approach is verified by means of preliminary numerical simulations in which buried target are illuminated by a set of incident waves in a noisy environment.

I. INTRODUCTION

Electromagnetic prospection of buried targets has acquired a great importance in several applicative fields, such civil engineering and archeology [1–5]. In recent years, several techniques have been devised, ranging from radar-based approaches to inverse-scattering-based tomographic imaging methods [6–15]. However, the development of efficient imaging algorithms, able to mitigate the drawbacks of existing approaches, is still a challenging task.

electromagnetic In prospection approaches, the electromagnetic field scattered by the target (when illuminated by a known source) is collected in a set of measurement points located above the air-ground interface and, when possible, in a borehole arrangement [15]. Assuming that the targets have cylindrical symmetries and that the illuminating field is transverse magnetic (TM), the relationship between such field samples and the dielectric properties of the investigated region can be modeled by using integral equations involving the Green's function for half-space configurations. One of the main problems is related to the ill-posedness of these equations, which requires the use of regularized inversion algorithms. Moreover, the overall relationship turns out to be nonlinear.

In the present work, an algorithm based on iterative linearization of the model and regularization in Banach spaces is considered. Such method is an extension to half-space configurations of the algorithm in [16], which has been proposed for free-space imaging of dielectric targets. In particular, after the local linearization is done, the obtained linear equation is solved through minimization of a cost function measured by l^p -norms, $1 . With respect to all the well know reconstruction algorithms, usually based on the classical Euclidean <math>l^2$ -norm, the present method in

 l^p Banach spaces is able to give a substantial reduction of the over-smoothing and ringing effects in the restored solution. Basically, due to the geometrical properties of the l^p Banach spaces, the method could allow to obtain a better localization and shaping of targets buried in the soil.

The mathematical formulation of the approach, both for the inverse scattering problem and the Banach space reconstruction algorithm, is detailed in Section II. Numerical evidences are provided in Section III, whereas some conclusions are briefly drawn in Section IV.

II. MATHEMATICAL FORMULATION

A. Electromagnetic inverse problem

Let us consider an infinite dielectric cylinder buried in a lossy soil. The target is illuminated by a TM electromagnetic field and the scattered electric field is collected by a set of antennas. A $e^{j\omega t}$ time dependence is assumed and omitted in the following. The *z*-component of the scattered electric field in the measurement points, e_{scatt} , is related to the dielectric properties of the investigation area by means of the following Lippmann-Schwinger equation [1]

$$e_{scatt}(\mathbf{r}) = -k_b^2 \int_{D_{inv}} c(\mathbf{r}') e_{tot}(\mathbf{r}') g_{hs}(\mathbf{r}, \mathbf{r}') d\mathbf{r}'$$

$$= G_{data}^{HS}(ce_{tot})(\mathbf{r})$$
(1)

where $k_b = \omega \sqrt{\epsilon_b \mu_0}$ is the wavenumber in the soil (being ϵ_b the complex dielectric permittivity and μ_0 the vacuum magnetic permeability), D_{inv} is the investigation area, e_{tot} is the z-component of the total electric field inside D_{inv} , and $g_{hs}(\mathbf{r}, \mathbf{r}')$ is the half-space Green's function [11]. In (1), *c* is the contrast function, defined as

$$c(\mathbf{r}) = \frac{\epsilon(\mathbf{r})}{\epsilon_b} - 1 \tag{2}$$

being ϵ the complex dielectric permittivity in the investigation area. The electric field inside the investigation area can be written in terms of the contrast function by means of a second equation, i.e.,

$$e_{inc}(\mathbf{r}) = e_{tot}(\mathbf{r}) + k_b^2 \int_{D_{inv}} c(\mathbf{r}') e_{tot}(\mathbf{r}') g_{hs}(\mathbf{r}, \mathbf{r}') d\mathbf{r}'$$

$$= e_{tot}(\mathbf{r}) - G_{state}^{HS}(ce_{tot})$$
(3)

Equations (1) and (3) are combined together for obtaining a non-linear operator equation that can be formally written as

$$e_{scatt}(\mathbf{r}) = G_{data}^{HS} c (I - G_{state}^{HS} c)^{-1} e_{inc}(\mathbf{r}) = \mathcal{F}(c)(\mathbf{r})$$
(4)

where $\mathcal{F}: X \to Y$, being X and Y two fixed Banach spaces, is a nonlinear operator mapping the contrast function *c* into the scattered field e_{scatt} .

B. Banach space inversion procedure

The nonlinear inverse scattering problem can be formulated as follows: Given the scattered field $e_{scatt} \in Y$, find the contrast function $c \in X$ by solving the nonlinear functional equation

$$\mathcal{F}(c) = e_{scatt} \tag{5}$$

After a suitable discretization of the functional equation, based on piecewise constant basis functions, we search for (an approximation of) the contrast function c by means of the following iterative scheme based on an inexact-Newton method in Banach spaces [16].

Starting from an initial guess $c_0 \in \mathbb{C}^N$ (e.g., a simple "empty" domain $c_0 = 0$ can be used if no a-priori information is available), for k = 0, 1, 2, ..., the iteration $c_{k+1} = c_k + \delta_k \in \mathbb{C}^N$ is computed, being $\delta_k \in \mathbb{C}^N$ an l^p –regularized solution of the linear equation

$$F_k \delta_k = b_k \tag{6}$$

where $b_k = e_{scatt} - F(c_k) \in \mathbb{C}^Q$. Here $F_k: \mathbb{C}^N \to \mathbb{C}^Q$ denotes the Fréchet derivative of the operator $F: \mathbb{C}^N \to \mathbb{C}^Q$ at point c_k , that is, its best local linear approximation in a neighborhood of c_k . The iterations will stop until a predefined stopping criteria is fulfilled, such as, for instance, the discrepancy principle [17].

In the discrete version of the model, the Banach spaces X and Y become the well known $l^p(\mathbb{C}^N)$ or $l^p(\mathbb{C}^Q)$ Banach spaces. Here N denotes the number of elements of the discretization of the contrast function c, that is, the number of pixels of the investigation domain, and Q denotes the total number of measurements of the scattered field, related to all the different illuminations and different locations of both emitting and receiving antennas.

The peculiarity of the method is the special computation of any linear equation (6), that is, the application of an l^p Banach space regularization algorithm, with a fixed value $1 . In particular, we consider an iterative method which minimizes the square of the <math>l^p$ -norm of the residual, that is, which minimizes the following residual cost functional $R_p: \mathbb{C}^N \to \mathbb{R}$ defined as

$$R_p(\delta) = \frac{1}{2} \|F_k \delta - b_k\|_p^2, \tag{7}$$

where the square of the l^p -norm is $||v||_p^2 = \left(\sum_{i=1}^Q |v_i|^p\right)^{2/p}$.

We notice that, for p = 2, the residual cost function is the classical least square functional in the l^2 Hilbert space. In this case, the basic iterative minimization scheme is the steepest descent method with fixed step length, namely the Landweber method, defined as follows: for $l = 0, 1, 2, ..., l_{max}$ (or until a suitable stopping rule is satisfied)

$$\delta_{k,l+1} = \delta_{k,l} - \tau F_k^* \big(F_k \delta_{k,l} - b_k \big) \tag{8}$$

with the initial guess $\delta_{k,0} = 0$. Here $0 < \tau < 2/||F_k||^2$ is a fixed relaxation parameter, which always guarantees the convergence of the iterations $\delta_{k,l+1}$, as *l* goes to infinity, towards the minimum of R_2 , which is the solution δ_k of the linear equation (6). The classical Landweber method [18] for l^2 is thus generalized to the minimization of the l^p cost functional (7) as follows

$$\delta_{k,l+1} = J_N^{p*} \left(J_N^p(\delta_{k,l}) - \tau F_k^* J_Q^p(F_k \delta_{k,l} - b_k) \right)$$
(9)

for $l = 0, 1, 2, ..., l_{max}$, with again the initial guess $\delta_{k,0} = 0$. The function $J_N^p: \mathbb{C}^N \to \mathbb{C}^N, J_Q^p: \mathbb{C}^Q \to \mathbb{C}^Q$ and $J_N^{p^*}: \mathbb{C}^N \to \mathbb{C}^N$ are the discrete versions of the so-called duality maps [19] and are in general defined as follows

$$J_{M}^{r}(v) = \|v\|_{r}^{2-r} \begin{pmatrix} |(v)_{1}|^{r-1}\operatorname{sign}(v)_{1} \\ |(v)_{2}|^{r-1}\operatorname{sign}(v)_{2} \\ \vdots \\ |(v)_{M}|^{r-1}\operatorname{sign}(v)_{M} \end{pmatrix},$$
(10)

with $\operatorname{sign}(v) = \begin{cases} e^{j \operatorname{arg}(v)} & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases}$ and p^* the Holder conjugate of the value p, that is, $\frac{1}{p} + \frac{1}{p^*} = 1$.

As a general comment, with respect to classical least square regularization (that is, regularization in the l^2 norm of any Hilbert space), the proposed regularization methods with l^p norm in Banach space allows usually to obtain a better localization of targets and less artifacts and ringing effects in the images of the retrieved objects [16].

III. PRELIMINARY NUMERICAL RESULTS

The considered approach have been preliminary tested by using simulated data. Similarly to [13], a "mixed" measurement configuration is used. The antennas are located both on the air-ground interface and in two boreholes located next to the investigated area. In particular, their positions are the following

 $\mathbf{r}^{(n)}$

$$=\begin{cases} (-2\lambda_0/3, -4\lambda_0/3 + 2(n-1)\lambda_0/21), n = 1, \dots, 15\\ (-2\lambda_0/3 + 2(n-15)\lambda_0/21, 0), & n = 16, \dots, 28\\ (2\lambda_0/3, -2(n-29)\lambda_0/21), & n = 29, \dots, 43 \end{cases}$$
(1)

where λ_0 is the wavelength in air. A subset of the available antennas illuminates the investigated area. When an antenna works as a transmitter, the remaining M = 42 collect the scattered electromagnetic field. For the case reported in the present paper, S = 7 antennas, located at positions denoted by the indexes n = 7(s - 1) + 1, s = 1, ..., S, are used as transmitter. The total number of measurements of the scattered field is $Q = 7 \times 42$.

The working frequency is 300 MHz. A dry sandy soil, modeled as a homogeneous material with dielectric permittivity $\epsilon_b = 4\epsilon_0$ and electric conductivity $\sigma_b =$ 0.01 S/m, is assumed. The investigation domain is a square area of side λ_0 and center $(0, -\lambda_0/2)$, which has been discretized into $N = 30 \times 30$ subdomains. The input scattered field data has been simulated by using a numerical code based on the method of moments. In order to avoid an inverse crime, a different discretization has been used. Moreover, the computed values have also been corrupted by an additive Gaussian noise with zero mean value and variance corresponding to a signal-to-noise ratio of 25 dB.

The target is composed by a circular dielectric object of radius $a = 0.15\lambda_0$, dielectric permittivity $\epsilon_{obj} = 6\epsilon_0$, and whose center is located at $\mathbf{r}_{obj} = (-0.15\lambda_0, -0.5\lambda_0)$. The inversion algorithm has been executed with the following parameters: Maximum number of inner, i.e., Landweber in Banach spaces, iterations of (9), $l_{max} = 5$; maximum number of outer, i.e., Gauss Newton, iterations of (6), $k_{max} = 100$. Moreover, the inversion is started from a void investigation area (with dielectric properties equal to those of the soil).

The quality of the reconstruction has been evaluated by means of the following mean relative errors

$$e_{\begin{cases} inv\\bg\\obj \end{cases}} = \frac{1}{N_{\begin{cases} bg\\obj \end{cases}}} \sum_{n} \frac{|\epsilon_n^{actual} - \epsilon_n^{rec}|}{|\epsilon_n^{actual}|}$$
(2)

where ϵ_n^{actual} and ϵ_n^{rec} are the values of the actual and reconstructed complex permittivity in the *n*-th subdomain of the investigation region, and the subscript *inv*, *bg*, and *obj* indicate that the relative quantities are evaluated by considering the subdomains belonging to the whole investigation area, the object, and the background, respectively.

The behaviors of such error measures versus the norm parameter are reported in Fig. 1. As can be seen, low values of p provide better reconstructions of the whole investigation area with respect to the case of standard Hilbert-space methods (corresponding to p = 2).

This fact is also confirmed by Fig. 2, which reports the reconstructed distributions of the relative dielectric permittivity in the investigation area for p = 1.4 (i.e., the case corresponding to the lowest error on the object) and p = 2.0 (i.e., the standard Hilbert approach). Finally, for completeness, the reconstructed dielectric profiles along a vertical line passing through the center of the cylinder are reported in Fig. 3 for p equal to 1.4, 2.0 and 2.3.



Fig. 1. Mean relative reconstruction error versus the norm parameter p.



Fig. 2. Reconstructed distributions of the relative dielectric permittivity with (a) p = 1.4 (Banach-based regularization) and (b) p = 2.0 (Hilbert-based regularization).



Fig. 3. Vertical cut (x = -0.15 m) of the reconstructed distributions of the relative dielectric permittivity with (a) p = 1.4, (b) p = 2.0, and (c) p = 2.3.

IV. CONCLUSION

A Banach-space-based approach for the electromagnetic imaging of buried objects has been considered in the present paper. The proposed method is based on the full non-linear formulation of the electromagnetic scattering problem for half spaces and makes use of an inexact-Newton linearization scheme. In such a scheme, any linearized system is solved by means of a regularization algorithm in Banach spaces, which minimizes the l^p -norms of the residual function. Solution of inverse problems in Banach spaces is an emerging mathematical framework that seems to exhibits good properties for the reconstruction of isolated objects (e.g., isolated materials in soil), thus resulting in a better localization and shaping. The reported numerical results, although still preliminary, show that the method is able to obtain quite good results, even in presence of lossy soil and complex objects in a noisy environment.

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