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An accurate near-field solver for scattering from cylinders with corners

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***Abstract*—This paper is about scattering of E-waves (also referred to as TM-waves) from an infinitely long, perfectly conducting cylinder with piecewise smooth boundary. The problem is formulated as an integral equation and is solved by Nyström's method in combination with a method referred to as Recursively Compressed Inverse Preconditioning (RCIP) that efficiently handles the corners. In the numerical example an incident plane E-wave is scattered from a cylinder with one corner. We present results for kd up to 1000, where d is the diameter of the cylinder and k is the wavenumber. Even for such a large cylinder we get at least 13 digits of accuracy for the electric field everywhere outside the cylinder.**

I. Introduction

Scattering from two-dimensional structures, like infinitely long cylinders, are of interest in, e.g. wave propagation in waveguides, photonics band-gap structures, substrate integrated waveguides, but also for elongated objects, like wings. In this paper we present an efficient method that can handle scattering from cylinders with an arbitrary number of corners. Close to corners the surface quantities that are used to represent the solution often have complicated asymptotics, which can be solved by introducing a finer mesh close to the corners. If this is not done with great care the number of unknowns grows rapidly with the number of corners and the accuracy deteriorates. In this paper we apply a method referred to as Recursively Compressed Inverse Preconditioning (RCIP). It was originally developed for static problems but in a recent paper [1] it was shown that it also can handle Helmholtz equation equally well. The method is described in more detail in [1] and in the tutorial [2]. In this paper we restrict ourselves to the E-wave case whereas both the E-wave and H-wave, also called TE-waves, cases are treated in [1] and [2].

We give examples where the scattered electric field from a cylinder with one corner and with a diameter of up to 160 wavelengths is obtained with 14 digits of accuracy almost everywhere outside the cylinder. This success is achieved by

1) choosing a suitable integral representation of the scattered field in terms of an unknown layer density
2) formulating the scattering problem as a Fredholm second kind integral equation with operators that are compact away from the corners
3) discretizing using a Nyström scheme and a mix of composite Gauss–Legendre quadrature and high-order analytic product rules
4) modifying the discretized integral equation so that the new unknown, a transformed layer density, is piecewise smooth
5) solving the resulting well-conditioned linear system iteratively for the transformed layer density
6) partially reconstructing the original layer density from the transformed layer density
7) evaluating the scattered field from a discretization of its integral representation which, again, relies on a mix of composite Gauss–Legendre quadrature and high-order analytic product rules

While some steps in this scheme are standard, step 4, 6, and 7 are unique to the recently developed RCIP method. Conceptually, step 4 and 5 correspond to applying a fast direct solver [3] locally to regions with troublesome geometry and then applying a global iterative method. This gives us many of the advantages of fast direct methods, for example the ability to deal with certain classes of operators whose spectra make them unsuitable for iterative methods. In addition, this approach is typically much faster than using only a fast direct solver.

II. Formulation of the problem

We consider in-plane E-waves scattered by an infinitely long bounded perfectly conducting cylinder with a piecewise smooth boundary Γ. The region outside the object is denoted Ω_{ex}, the time dependence is $e^{-\mathrm{i}\omega t}$ and $\boldsymbol{r} = (x, y)$.

A. Helmholtz equation and integral equation

We let the electric field be parallel to the cylinder, $\boldsymbol{E}(\boldsymbol{r}) = \hat{\boldsymbol{z}}U(\boldsymbol{r})$, and let $U(\boldsymbol{r}) = U_{\text{inc}}(\boldsymbol{r}) + U_{\text{sca}}(\boldsymbol{r})$. The scattered field $U_{\text{sca}}(\boldsymbol{r})$ satisfies the following exterior Dirichlet problem:

$$\nabla^2 U_{\text{sca}}(\boldsymbol{r}) + k^2 U_{\text{sca}}(\boldsymbol{r}) = 0,\ \boldsymbol{r} \in \Omega_{\text{ex}} \tag{1}$$

$$U_{\text{sca}}(\boldsymbol{r}) = -U_{\text{inc}}(\boldsymbol{r}),\ \boldsymbol{r} \in \Gamma \tag{2}$$

$$\lim_{|\boldsymbol{r}|\to\infty} \left(\frac{\partial}{\partial r} - \mathrm{i}k \right) U_{\text{sca}}(\boldsymbol{r}) = 0. \tag{3}$$

For , $\boldsymbol{r} \in \Omega_{\text{ex}}$ we write the solution as the combined integral representation [4, eq. (3.25)].

$$U_{\text{sca}}(\boldsymbol{r}) = \int_\Gamma \frac{\partial \Phi_k(\boldsymbol{r},\boldsymbol{r}')}{\partial \nu_{r'}} \rho(\boldsymbol{r}')\mathrm{d}\ell' - \mathrm{i}\frac{k}{2}\int_\Gamma \Phi_k(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}\ell' \tag{4}$$

where $\Phi_k(\boldsymbol{r},\boldsymbol{r}') = \dfrac{\mathrm{i}}{4}H_0^{(1)}(k|\boldsymbol{r}-\boldsymbol{r}'|)$ is the free space Green function for the Helmholz equation in two dimensions, $H_0^{(1)}$ is the Hankel function of the first kind of order zero, and $\mathrm{d}\ell$ is an element of arc length. The index k indicates that the quantity or function depends on the wavenumber $k = \omega/c$. Insertion of (4) into (2) gives the integral equation for the layer density $\rho(\boldsymbol{r})$

$$(I + K_k - \mathrm{i}\frac{k}{2}S_k)\rho(\boldsymbol{r}) = -2U_{\text{inc}}(\boldsymbol{r}),\ \boldsymbol{r} \in \Gamma, \tag{5}$$

where

$$K_k\rho(\boldsymbol{r}) = 2\int_\Gamma \frac{\partial \Phi_k(\boldsymbol{r},\boldsymbol{r}')}{\partial \nu_{r'}}\rho(\boldsymbol{r}')\mathrm{d}\ell' \tag{6}$$

$$S_k\rho(\boldsymbol{r}) = 2\int_\Gamma \Phi_k(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}\ell'. \tag{7}$$

The second term on the right hand side in (4) corresponds to the term $\mathrm{i}\dfrac{k}{2}S_k$ in (5) and is added in order to ensure a unique solution for all k. The equation (5) is often referred to as an indirect combined field integral equation (ICFIE).

III. Numerical scheme

The numerical solver used in this paper takes its starting point in a Fredholm second kind integral equation with integral operators that are compact away from boundary singularities and whose unknown quantity is a layer density representing the solution to the original problem. The integral equation is discretized using a Nyström scheme and composite Gauss–Legendre quadrature. At the heart of the solver lies a method called Recursively Compressed Inverse Preconditioning (RCIP). It modifies the kernels of the integral operators so that the layer density becomes piecewise smooth and simple to resolve by polynomials. Loosely speaking one can say that RCIP makes it possible to solve elliptic boundary value problems in piecewise smooth domains as cheaply and accurately as they can be solved in smooth domains. The RCIP method originated in 2008 [5]. In a series of papers it has been extended and successfully applied to electrostatic and elastostatic problems which, at first glance, might seem outright impossible. For example, the effective conductivity of a high-contrast conducting checkerboard with a million randomly placed squares in the unit cell was computed on a regular workstation with a relative accuracy of 10^{-9} [6]. A new record has been established for the three-dimensional problem of determining the capacitance of the unit cube – 13 digits compared to the seven digits that were previously known [7].

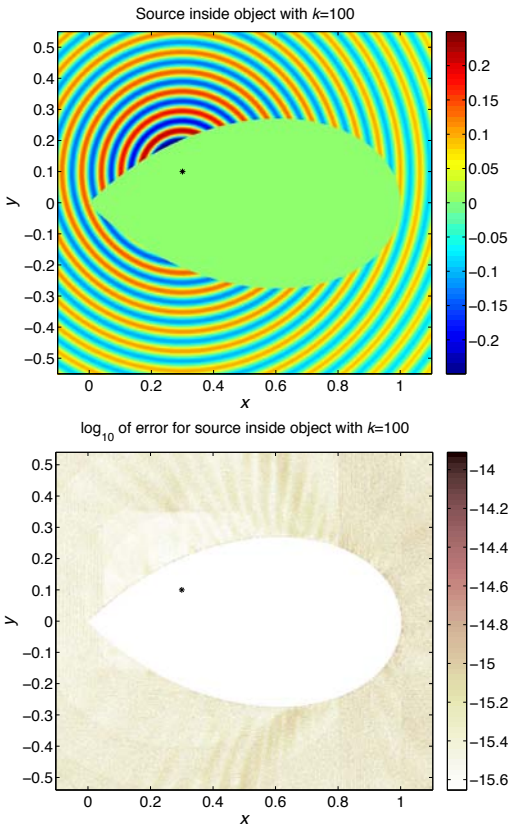

Fig. 1. This is a convergence test. The upper figure shows the electric field from a line source at $\boldsymbol{r} = (0.3, 0.1)$ inside Γ. It is obtained by solving the boundary value problem with boundary values on Γ, given by the line source. The lower figure shows the error compared to the exact solution.

A. Convergence and error estimates

Our solver shows a stable behavior. This means that the solution converges rapidly with coarse mesh refinement up until a point beyond which no further improvement occurs. Actually, beyond this optimal point there will be a slow decay in the quality of the solution, due to accumulated roundoff error. The precise location of the optimal point is hard to determine *a priori*. It depends on the geometry, on the boundary conditions, and on the wave number. The optimal point is determined experimentally in the numerical examples of Section IV.

We have estimated the accuracy in our solutions $U(\boldsymbol{r})$ rather thoroughly. The lower plot of Figure 1 shows the error plots for an exterior problem. It is achieved by generating the boundary conditions on Γ via a line source at $\boldsymbol{r} = (0.3, 0.1)$ inside Γ so that the exact solution is known. The upper field plot in Figure 1 shows the field generated from the line source by solving the boundary value problem. The lower plot shows that we have more than 14 digits accuracy in our solution. In the plane-wave scattering examples of Section IV-A, below, no exact results are known. Therefore we proceed as follows: we first compute a solution $U(\boldsymbol{r})$ using a number of coarse panels on Γ deemed sufficient for resolution. Then we increase this number with 50 % and solve again. The difference between

the resolved value of $U(\boldsymbol{r})$ and the overresolved value of $U(\boldsymbol{r})$ is used as an indirect pointwise error estimate. Yet an indirect method to estimate the (overall) precision in the computations is by comparing the scattering cross section computed from its definition with its value obtained via the optical theorem (at infinity). See, further, [1]. As it turns out, the various error estimates agree to 15 digits.

IV. Numerical examples

We shall now solve (5) for the unknown density $\rho(\boldsymbol{r})$, using using a Nyström scheme with a composite Gauss–Legendre quadrature, and the RCIP solver, and then evaluate the scattered field of (4). We restrict the numerical examples to scattering from an infinite straight cylinder with boundary Γ described by

$$\boldsymbol{r}(t) = \sin(\pi t)\left(\cos\left((t-0.5)\frac{\pi}{2}\right), \sin\left((t-0.5)\frac{\pi}{2}\right)\right), \quad (8)$$

where $t \in [0,1]$, and to the incident plane wave $U_{\text{inc}}(\boldsymbol{r}) = e^{\mathrm{i}ky}$. The object parameterized in (8) has a corner with opening angle $\theta = \pi/2$ at $\boldsymbol{r} = 0$ and a diameter $d = 1$, in arbitrary length units, so that $kd = k$. The examples cover sizes from $kd = 1$ up to $kd = 1000$. We have seen that at $kd = 1000$ the frequency is high enough such that the uniform theory of diffraction theory can be applied. All numerical examples are executed in MATLAB on a workstation equipped with an IntelXeon E5430 CPU at 2.66 GHz and 32 GB of memory.

A. Near field

A criterion for a powerful method is that it should be able to calculate the electric field everywhere in Ω_{ex}. Figure 2 shows the total electric field for the E-wave in the vicinity of the scattering object and the corresponding errors. The scattering object itself appears in green color in the upper image and in white color in the image below. The number of spatial points in each image is 10^6. It is encouraging to see that the accuracy is high even close to the boundary and, in particular, close to the corner. The integrals in (4) are often thought of as difficult to evaluate close to the boundary due to the singularities in the Hankel functions when $\boldsymbol{r}' = \boldsymbol{r}$. However, the present method circumvents these problems using a high-order analytic quadrature.

In Figures 2 a), e) the real part of the total electric field $U(\boldsymbol{r})$ for the E-wave case is plotted for $kd = 10$ and 1000. To capture the diffraction pattern in the vicinity of the corner, the field is plotted in a rectangular region with side length proportional to $1/k$ and center at the tip of the corner. At $kd = 10$ the error is very small, as seen from Figure 2 b). The errors increase slightly with kd but even at $kd = 1000$ we get 14 digits or better almost everywhere, as depicted in Figure 2 f).

V. Conclusion

We show that scattering of a plane wave from an infinitely long cylinder with piecewise smooth boundary and with an finite, but arbitrary, number of corners can be efficiently treated by a numerical method that utilizes the Nyström method

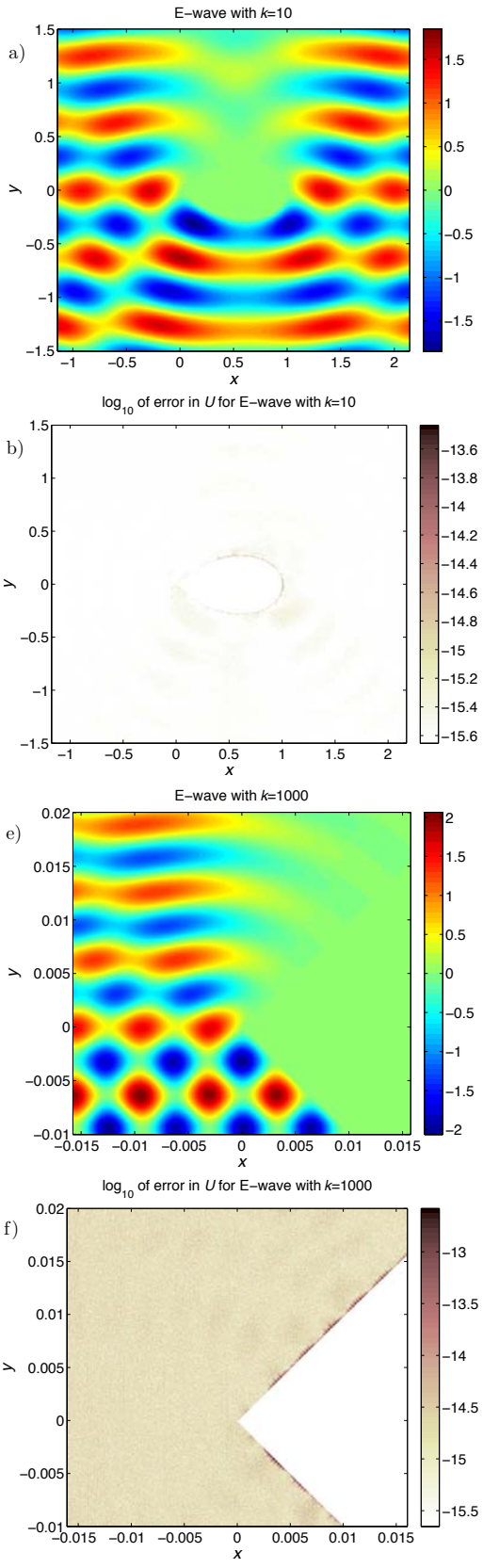

Fig. 2. a), e) show $\Re\{U(\boldsymbol{r})\}$ for a plane E-wave $U_{\text{inc}}(\boldsymbol{r}) = e^{\mathrm{i}ky}$ incident on the perfectly conducting cylinder with boundary Γ given by (8). b), f) show absolute errors.

and RCIP. When we here apply the RCIP method to the Helmholtz equation and the exterior Dirichlet problem we do this in a two-dimensional setting. We consider scattering of time-harmonic E-waves from an infinitely long perfectly conducting cylinder. Scattering problems are harder to solve than electrostatic problems, all other things held equal. Planar problems provide a good testing ground prior to a move up to three dimensions [8]. As we have seen, the transition from Laplace's equation to the Helmholtz equation is surprisingly straightforward and the results, presented in Section IV, are as good as the ones obtained for electrostatics.

ACKNOWLEDGMENT

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