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Adjoint Sensitivity Analysis with Analytical Shape Derivatives of the EFIE System

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***Abstract*—We derive analytical shape integral formulas for the shape derivatives of the system matrix arising from the Rao-Wilton-Glisson (RWG) discretized electric field integral equation.**

Using these formulas and the adjoint variable method (AVM), we inspect sensitivity of input reflection coefficient of a certain UWB dipole antenna with respect to infinitesimal variations of the computational mesh.

The sensitivity obtained with the analytical formula is compared against one calculated with the finite difference approximation.

I. Introduction

Antenna shape optimization by gradient methods has become increasingly feasible even with adequately powerful desktop computers during the last decade. The forward problem of solving the input admittance of a moderately simple antenna consisting of perfect electric conducting (PEC) material is easily solvable with electric field integral equation (EFIE) within few seconds to minutes.

The shape sensitivity of the input admittance has been computed traditionally with adjoint variable methods [1], [2] employing finite difference formulas [1], [3]. It is unclear if the finite difference (FD) formulas give the right results and, furthermore, the FD approach is rather cumbersome and computationally expensive. Also automatic differentiation has been used in the past [4]. Both aforementioned methods give good results, but they don't tell anything about local behavior of the shape differentials in an analytical sense.

Analytical shape derivatives for planar meshes have been previously studied in [5]. In the present work, however, the geometry and shape variations are not restricted to planar ones. Furthermore, the change of variables is employed together with the Piola transformation [6] in contrast to the flux-transport theorem used in [5].

The core results of this work are simple analytical formulas for the shape derivatives of the system matrix arising from the EFIE discretized with RWG basis functions. In addition, we discuss how the shape derivatives can be computed together with the original system matrix and we point out that the increase in computation time is small.

II. Preliminaries

The boundary current $\mathbf{J}$ of a PEC object D with boundary $\Gamma = \partial D$ satisfies [7]–[11]:

Find $\mathbf{J}$ *s.t.*

$$\frac{i}{\omega\epsilon}\int_\Gamma \operatorname{div}\mathbf{u}(\mathbf{r})\mathcal{S}(\operatorname{div}\mathbf{J})(\mathbf{r})\mathrm{d}\mathbf{r} - i\omega\mu\int_\Gamma \mathbf{u}(\mathbf{r})\cdot\mathcal{S}(\mathbf{J})(\mathbf{r})\mathrm{d}\mathbf{r} = \int_S \mathbf{u}(\mathbf{r})\cdot\mathbf{E}_p(\mathbf{r})\mathrm{d}\mathbf{r} \quad \forall\mathbf{u}. \tag{1}$$

Here $\mathcal{S}$ is the single layer operator defined by

$$\mathcal{S}(\mathbf{J})(\mathbf{r}) = \int_\Gamma G_k(\mathbf{r}-\mathbf{r}')\mathbf{J}(\mathbf{r}')\mathrm{d}\mathbf{r}', \tag{2}$$

where G_k is given by (time harmonic sign convention $e^{-i\omega t}$)

$$G_k(\mathbf{r}) = \frac{e^{ik|\mathbf{r}|}}{4\pi|\mathbf{r}|}. \tag{3}$$

Let us suppose that the boundary Γ is admits triangulation $\mathcal{T} = (T_p)_{p=1}^N$ where all the triangles are flat. The Equation (1) is discretized with modified RWG basis functions given by

$$\mathbf{u}_p^n(\mathbf{r}) = \frac{1}{2A_p}\left(\mathbf{r}-\mathbf{p}^{n+2}\right), \tag{4}$$

where A_p is the area of the triangle T_p and $\mathbf{p}^{n+2}$ is the opposing vertex to the edge having local index n. It should be noted that the scaling by edge length is omitted here, which makes these basis functions exactly the lowest order Raviart-Thomas basis functions [6].

When computing the elements of the system matrix arising from formulation (1) one arrives to the following double integrals

$$I_1 = \int_{T_q}\mathbf{v}(\mathbf{r})\cdot\int_{T_p}G_k(\mathbf{r}-\mathbf{r}')\mathbf{u}(\mathbf{r}')\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r} \tag{5}$$

$$I_2 = \int_{T_q}\operatorname{div}_{T_q}\mathbf{v}(\mathbf{r})\int_{T_p}G_k(\mathbf{r}-\mathbf{r}')\operatorname{div}_{T_p}\mathbf{u}(\mathbf{r}')\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}, \tag{6}$$

where $\mathbf{u}$ and $\mathbf{v}$ are RWG basis functions on T_p and T_q, respectively, and div_T denotes the surface divergence on the surface patch T.

Let us denote the linear system arising from the method of moments discretization by

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{7}$$

In this work, the system matrix $\mathbf{A}$ and $\mathbf{x}$ depend upon a small real parameter s and this dependence is denoted by $\mathbf{A}_s$ and $\mathbf{x}_s$. However, when $s = 0$, it is omitted.

A. The Piola Transformation

Let F_p be the affine map which maps the reference triangle

$$\widehat{T} = \{(\xi,\eta) :\ 0 < \xi + \eta < 1 \text{ and } \xi,\eta > 0\} \tag{8}$$

to the triangle T_p. We note that the determinant of its Jacobian F_p' is given by

$$\det F_p' = |F_p'\mathbf{e}_\xi \times F_p'\mathbf{e}_\eta|. \tag{9}$$

Here the unit vectors of the coordinate directions ξ and η are denoted by $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$, respectively.

The Piola transformation [6] $\mathcal{P}_{F_p}$ associated with F_p is defined by

$$\mathcal{P}_{F_p}\,\widehat{\mathbf{u}} \circ F_p = \frac{1}{\det F_p'} F_p'\widehat{\mathbf{u}}. \tag{10}$$

It maps the RWG basis functions on $\widehat{T}$ to RWG basis functions on T_p. Furthermore, it has the following property [6]

$$\operatorname{div}_{T_p} \mathcal{P}_{F_p}\,\widehat{\mathbf{u}} = \frac{1}{\det F_p'} \left(\operatorname{div}_{\widehat{T}}\widehat{\mathbf{u}}\right) \circ F_p^{-1}, \tag{11}$$

where the surface divergences on $\widehat{T}$ and T_p are denoted by $\operatorname{div}_{\widehat{T}}$ and div_{T_p}, respectively.

B. Adjoint Variable Method

Let use briefly review the adjoint variable method (AVM) in order to fix notations and to motivate the computation of the derivatives of $\mathbf{A}_s$. For a more comprehensive treatment on the matter we refer to [1], [2], [12]. Early signs of AVM can also be found in [13].

Suppose $\mathbf{x}_s$ is a solution to some single port antenna computation problem with an excitation given by vector $\mathbf{b}$. The input admittance is given by $Y(\mathbf{x}_s) = \mathbf{x}_s^T\mathbf{b}$ and its derivative with respect to s satisfies

$$\frac{d}{ds}Y(\mathbf{x}_s) = \left(\frac{\partial}{\partial \mathbf{x}}Y\right)^T \frac{d\mathbf{x}_s}{ds} \tag{12}$$

The state equation is dictated by the MoM system

$$\mathbf{A}_s\mathbf{x}_s = \mathbf{b}. \tag{13}$$

Differentiating (13) on both sides with respect to s we obtain

$$\mathbf{A}\frac{d\mathbf{x}_s}{ds} = -\frac{d\mathbf{A}_s}{ds}\mathbf{x} \tag{14}$$

Introducing the adjoint problem

$$\mathbf{A}^T\gamma = \frac{\partial}{\partial \mathbf{x}}Y(\mathbf{x}) = \mathbf{b}, \tag{15}$$

we arrive to

$$\frac{dY}{ds} = -\gamma^T\frac{d\mathbf{A}_s}{ds}\mathbf{x}. \tag{16}$$

However, since the EFIE system matrix is symmetric, we obtain the usual [4] result

$$\frac{dY}{ds} = -\mathbf{x}^T\frac{d\mathbf{A}_s}{ds}\mathbf{x}. \tag{17}$$

III. The Derivative Formulas

In this section we shall derive the analytical derivative formulas. We start by inspecting small variations of triangles.

Let us denote the mapping that moves one vertex of the triangulation $\mathcal{T}$ by an amount of s to the direction $\boldsymbol{\tau}$ by Φ_s. It holds that

$$\Phi_s(\mathbf{r}) = \mathbf{r} + s\boldsymbol{\tau}\lambda_m(\mathbf{r}), \tag{18}$$

where λ_m is the first order nodal shape function associated with the vertex m.

The Jacobian Φ_s' is given by

$$\Phi_s' = \mathbf{I} + s\boldsymbol{\tau}\nabla\lambda_m, \tag{19}$$

where $\mathbf{I}$ is the identity dyad and $\nabla\lambda_m$ is interpreted as a row vector.

By denoting $T_p^s = \Phi_s(T_p)$ and F_p^s the affine mapping that takes $\widehat{T}$ to T_p^s, where $T_p \in \mathcal{T}$, it holds that

$$\Phi_s \circ F_p = F_p^s, \tag{20}$$

thus, the determinant of the Jacobian Φ_s' satisfies

$$\det \Phi_s' = \frac{\det F_p^{s\prime}}{\det F_p'}. \tag{21}$$

It should be noted that the determinant defined this way is not equivalent to the one obtained by extending Φ_s to an open neighborhood of T_p except when λ_m is extended by constant in the normal direction of T_p.

Now the Formula (10) can be used to define the Piola transformation associated with Φ_s without any changes.

It turns out that this maps again the RWG basis functions of T_p to those of T_p^s.

Let us denote

$$I_1^s = \int_{T_q^s} \mathbf{v}_{T_q^s}(\mathbf{r}) \cdot \int_{T_p^s} G_k(\mathbf{r}-\mathbf{r}')\mathbf{u}_{T_p^s}(\mathbf{r}')\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r} \tag{22}$$

and

$$I_2^s = \int_{T_q^s} \operatorname{div}_{T_q^s}\mathbf{v}_{T_q^s}(\mathbf{r}) \cdot \int_{T_p^s} G_k(\mathbf{r}-\mathbf{r}')\operatorname{div}_{T_p^s}\mathbf{u}_{T_p^s}(\mathbf{r}')\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}. \tag{23}$$

In the following, let us denote $\mathbf{u} = \mathbf{u}_{T_p}$ and $\mathbf{v} = \mathbf{v}_{T_q}$ for readability. By a change of variables we obtain

$$\begin{aligned} I_1^s &= \int_{T_q}\int_{T_p} \left(\Phi_s'(\mathbf{r})\frac{\mathbf{v}(\mathbf{r})}{\det \Phi_s'(\mathbf{r})}\right) \cdot \left(\Phi_s'(\mathbf{r}')\frac{\mathbf{u}(\mathbf{r}')}{\det \Phi_s'(\mathbf{r}')}\right) \\ &\qquad G_k\left(\Phi_s(\mathbf{r}) - \Phi_s(\mathbf{r}')\right)\det\Phi_s'\det\Phi_s'\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r} \\ &= \int_{T_q}\int_{T_p} \left(\Phi_s'(\mathbf{r})\mathbf{v}(\mathbf{r})\right)\cdot\left(\Phi_s'(\mathbf{r}')\mathbf{u}(\mathbf{r}')\right) \\ &\qquad G_k\left(\Phi_s(\mathbf{r}) - \Phi_s(\mathbf{r}')\right)\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r} \end{aligned} \tag{24}$$

and

$$I_2^s = \int_{T_q}\int_{T_p} \frac{\mathrm{div}_{T_q}\mathbf{v}(\mathbf{r})}{\det \Phi_s'(\mathbf{r})} \frac{\mathrm{div}_{T_p}\mathbf{u}(\mathbf{r}')}{\det \Phi_s'(\mathbf{r}')} G_k\left(\Phi_s(\mathbf{r}) - \Phi_s(\mathbf{r}')\right) \det \Phi_s'(\mathbf{r}) \det \Phi_s'(\mathbf{r}')\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}$$

$$= \int_{T_q}\int_{T_p} \mathrm{div}_{T_q}\mathbf{v}(\mathbf{r})\mathrm{div}_{T_p}\mathbf{u}(\mathbf{r}') G_k\left(\Phi_s(\mathbf{r}) - \Phi_s(\mathbf{r}')\right)\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}. \quad (25)$$

The derivatives of these expressions with respect to s are given by

$$\frac{\partial}{\partial s} I_1^s = \int_{T_q}\int_{T_p} \mathbf{v}(\mathbf{r}) \cdot \left((\boldsymbol{\tau}\nabla\lambda_m(\mathbf{r}))^T + \boldsymbol{\tau}\nabla\lambda_m(\mathbf{r}')\right) \cdot \mathbf{u}(\mathbf{r}')G_k(\mathbf{r}-\mathbf{r}') + \mathbf{v}(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r}')\boldsymbol{\tau} \cdot (\nabla G_k)(\mathbf{r}-\mathbf{r}')\left(\lambda_m(\mathbf{r}) - \lambda_m(\mathbf{r}')\right)\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r} \quad (26)$$

and

$$\frac{\partial}{\partial s} I_2^s = \int_{T_q}\int_{T_p} \mathrm{div}_{T_p}\mathbf{v}(\mathbf{r})\mathrm{div}_{T_q}\mathbf{u}(\mathbf{r}') \boldsymbol{\tau} \cdot (\nabla G_k)(\mathbf{r}-\mathbf{r}')\left(\lambda_m(\mathbf{r}) - \lambda_m(\mathbf{r}')\right)\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}. \quad (27)$$

For details, we refer to [14].

It is worth noting, that the integrals in (26) and (27) can be computed with, e.g., singularity subtraction methods [15] or singularity cancellation type methods [16], [17]. In this work we employed the former methods.

The integrands in Formulas (26) and (27) contain ∇G_k, which is of order R^{-2}, $R = |\mathbf{r} - \mathbf{r}'|$. However, it holds that $|\lambda_m(\mathbf{r}) - \lambda_m(\mathbf{r}')| < CR$, where C is finite and depends on the orientation of the triangles, thus for small R

$$|\boldsymbol{\tau} \cdot (\nabla G_k)(\mathbf{r}-\mathbf{r}')\left(\lambda_m(\mathbf{r}) - \lambda_m(\mathbf{r}')\right)| \le CR^{-1}. \quad (28)$$

The integrals (26) and (27) are not more singular than the original non-differentiated ones, but instead their polynomial order increases. Consequently, when computing the contribution arising from a well separated triangle pair to the derivative of the MoM system matrix, one has to pay close attention that the order of the numerical quadrature is sufficient.

IV. Adjoint sensitivity of dipole patch antenna

We study the sensitivity of the reflection coefficient ρ to $50\ \Omega$ transmission line of a bow-tie type antenna optimized in [12] by computing its derivative with respect to boundary variation shown in Figure 1. The sensitivity obtained by our analytical formula is then compared to one calculated with the first order forward finite difference formula.

The boundary variation field was constructed as follows. Let us denote the patch by $\Gamma \subset \mathbb{R}^3$ which has a boundary $\partial\Gamma$. The vertex next to $\partial\Gamma \cap \{(0,y) \in \mathbb{R}^2 : \ y > 0\}$ in clockwise direction is denoted by $\mathbf{p}_a$ and the vertex next to $\partial\Gamma \cap \{(0,y) \in \mathbb{R}^2 : \ y < 0\}$ in counterclockwise direction by $\mathbf{p}_b$. We assign to a part of the boundary in the right half plane a edge length parametrization $t \in [0, L]$ by assigning $t = 0$ to $\mathbf{p}_a$ and $t = L$ to $\mathbf{p}_b$. Here L is the length of the path obtained by traversing the boundary from $\mathbf{p}_a$ to $\mathbf{p}_b$ in clockwise direction.

At each boundary vertex $\mathbf{p}_m$ of $\mathcal{T}$ in the right half plane, we assign a vector

$$\boldsymbol{\tau}_m = \boldsymbol{\tau} \sin\left(\frac{k\pi t_m}{2L}\right), \quad (29)$$

where t_m is edge length coordinate of $\mathbf{p}_m$, $k = 1$ and $\boldsymbol{\tau} = \mathbf{e}_x$. The field is then mirrored to the vertices in the left half plane. This way we could easily construct more boundary variation fields by taking $k = 1, 2, \ldots, M$.

The matrix assembly code was constructed in such a way that, for each triangle the pair (T_p, T_q), it computes the original integrals (5) and (6) and their derivatives with respect to movement of each vertex to each direction $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$. The final derivative of the system matrix is then obtained by combining these partial derivatives. Thus, computing more derivatives of the system matrix does not significantly increase computation time. For instance, calculating partial derivatives of the system matrix with respect to 4, 8 or 16 parameters takes 5.0 s, 5.0 s or 5.5 s, respectively, whereas the assembly of the system matrix without derivatives lasts around 1 s.

The sensitivity of the reflection coefficient ρ with respect to the boundary variation field is shown in Figure 2. At 3 GHz a 1 mm variation would vary ρ by 0.05. The relative difference of the sensitivities given by the finite difference approximation and analytical formula is shown in Figure 3. We used step length of $h = 10^{-8}$ in the FD formula.

The reflection coefficient shown in Figure 4 is similar to the one computed in [12], where ρ was below 10 dB over the frequency range.

The antenna mesh in the present work was obtained by making a mesh over the image of the antenna in the electronic version of [12] with the DistMesh [18] mesh generator. That, together with the coarseness of the mesh explain the difference in the calculated reflection coefficient.

V. Conclusion

We discussed the analytical shape derivatives for the EFIE system matrix discretized with slightly unusual RWG functions. It turned out that such basis functions yield a very simple formula for the derivative of the system matrix. Furthermore, we demonstrated the applicability of the analytical formula by computing shape sensitivity of a previously optimized UWB dipole antenna and comparing it to the sensitivity obtained with the finite difference formula.

The studied UWB antenna is quite insensitive to variations in shape, yet the sensitivity is non-zero. The reason for this is that originally the antenna was optimized using a min-max procedure.

The main contribution of this work is the derivation of the analytical derivatives of the EFIE system matrix in such a form that existing quadratures can be easily used to compute them.

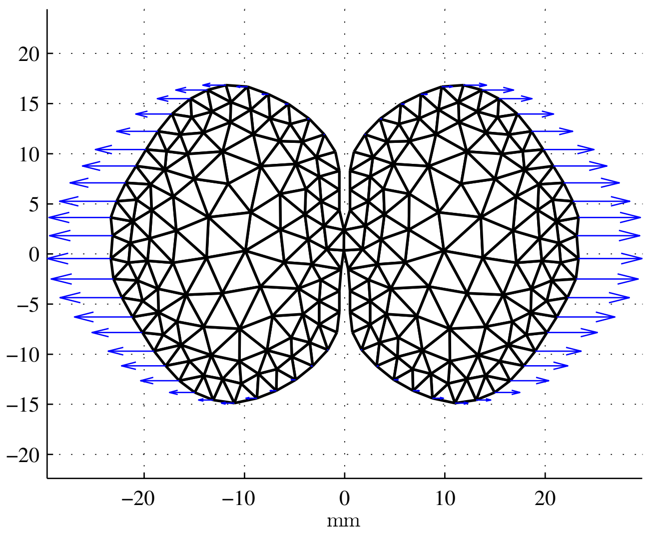

Figure 1. The boundary vector field to compute derivatives with respect to.

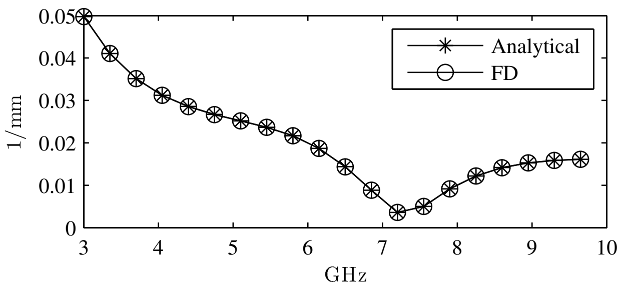

Figure 2. Sensitivity of reflection coefficient in mm with respect to boundary deformation field.

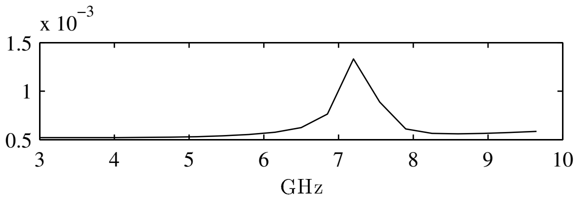

Figure 3. Relative difference of sensitivities obtained with difference formula and analytical one.

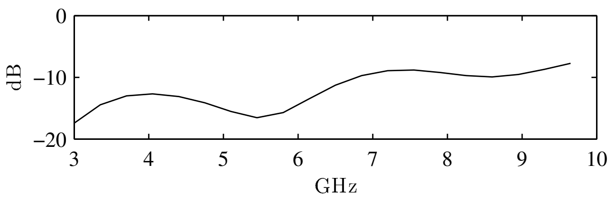

Figure 4. Reflection coefficient ρ

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