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# Exactly Solvable Chaos as Communication Waveforms

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**Abstract–**Recent developments enable the practical and beneficial use of chaos communications by control of symbolic dynamics. The key advance is the discovery of a new class of low-dimensional chaotic oscillators that enable simple encoding and coherent reception. These remarkable oscillators are provably chaotic, exhibit a symbolic dynamics with a generating partition, and admit an exact analytic solution as a linear convolution of symbols and a fixed basis function. For encoding information, the exact solution provides an analytic coding function to facilitate control of the system's symbolic dynamics. For reception, the existence of a fixed basis function enables a simple matched filter receiver for coherently detecting symbols. System performance exceeds other proposed chaos communications methods and approaches the theoretical limit of binary phase-shift keying (BPSK). Consequently the exactly solvable oscillators offer real advantages that justify using chaotic waveforms for high-bandwidth data communications.

## 1. Introduction

Over the last two decades, many have advocated the development of practical data communications using chaotic waveforms as information carriers. The motivations for such development and the proposed methods have been varied. Ideally, one would chose chaos for communications if, in some sense, a chaotic waveform is the optimal solution. That is, given practical and realistic constraints, the best method to communicate information from a source to destination uses a chaotic waveform.

In this short paper, we consider data communications subject to a realistic design constraint—correlation receiver performance using a simple, passive, analog matched filter. A correlation receiver is optimal for receiving discrete data in the presence of additive white Gaussian noise (AWGN) [1]. A passive analog filter offers low-cost, high-efficiency, durability and reliability. To meet the requirements imposed by these constraints, we show that an exactly solvable chaotic waveform is the best solution.

## 2. Communications Waveform

We suppose we have a random, information bit stream

$$s(t) = \sum_{m=-\infty}^{\infty} s_m \phi(t-m) \tag{1}$$

where

$$\phi(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

is a digital basis function (*i.e.*, square pulse) and each bit is represented by $s_m = \pm 1$. We consider this bit stream as a message signal and, in a certain sense, seek a best method to encode the message for transmission.

In many realistic environments, it is beneficial to use a correlation receiver to maximize the signal-to-noise ratio (SNR) and minimize the bit-error rate (BER). Ideally, we would implement a matched filter receiver; however, we also assume a strong motivation to avoid using a digital signal processor (DSP). That is, we require a simple solution due to power, cost, or other factors, such that the usual approach of digital sampling and processing are precluded.

As customary, we encode the message bits using a binary coded waveform

$$u(t) = \sum_{m=-\infty}^{\infty} s_m \cdot \rho(t-m) \tag{3}$$

where $\rho(t)$ is a fixed analog basis function. We choose this basis function so that it has a simple, analog matched filter. Assuming the waveform $u(t)$ is an electrical signal, a particularly simple matched filter is the *RLC* circuit shown in Figure 1. In this circuit, the input is $u(t)$ and the output is $x(t)$. This passive linear filter is modeled as

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \left(\omega^2 + \beta^2\right)(x-u) = 0 \tag{4}$$

where $\beta = T/2RC$, $\omega^2 + \beta^2 = T^2/LC$, and time is in units of the characteristic time $T$. By design, we assume the circuit components are selected such that $T = 1$,

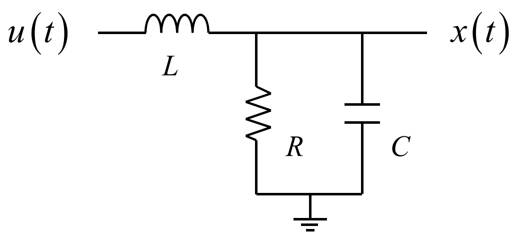


Figure 1. *RLC* filter proposed as a matched filter for the fixed basis function $\phi(t)$, with input signal $u(t)$ and output $x(t)$.

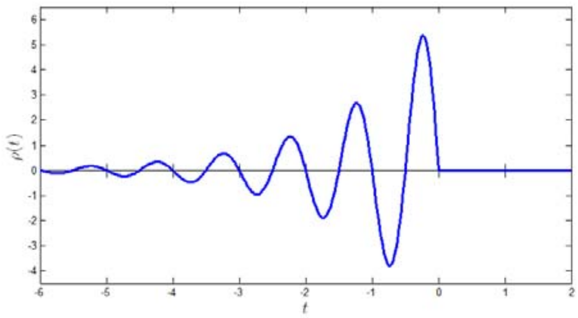


Figure 2. Basis function $\rho(t)$ corresponding to the simple matched filter circuit.

$\beta = \ln 2$, and $\omega = 2\pi$. Introducing an operator notation, we write

$$x(t) = \mathcal{M}_1 \circ u(t) \tag{5}$$

where $\mathcal{M}_1$ is a linear operator that denotes the operation of the matched filter.

We seek the basis function that corresponds to the matched filter shown in Figure 1. To this end, we recognize the impulse response of a matched filter for a real waveform is the time reversal of the waveform [1]. As such, we have

$$\mathcal{M}_1 \circ \delta(t) = \rho(-t) \tag{6}$$

where $\delta(t)$ is the impulse delta function. Explicitly, this equation requires solving the initial value problem

$$\frac{d^2\bar{\rho}}{dt^2} + 2\beta\frac{d\bar{\rho}}{dt} + \left(\omega^2+\beta^2\right)\left(\bar{\rho}-\delta(t)\right) = 0 \tag{7}$$

$$\bar{\rho}(-\infty) = 0, \quad \frac{d\bar{\rho}}{dt}(-\infty) = 0 \tag{8}$$

where $\bar{\rho}(t) = \rho(-t)$ is the time-reversed basis function. Solving this linear system yields

$$\rho(t) = \begin{cases} -\dfrac{\omega^2+\beta^2}{\omega}\cdot\sin(\omega t)\cdot 2^t, & t<0 \\ 0, & t\ge 0 \end{cases} \tag{9}$$

which is plotted in Figure 2. We note the basis function is not causal, since the waveform is nonzero as $t \to -\infty$. In conventional communications theory, the lack of causality is a stumbling block; however, we do not let this technicality dissuade us and we proceed anyway.

## 3. Chaos

For the basis function $\rho(t)$, it is significant that the waveform $u(t)$ is chaotic in the sense of Li-Yorke [2]. To support this claim, we define a return map from the waveform using the derivative evaluated at half-integer times and show it is conjugate to the Bernoulli shift map.

. To begin, we note

$$\frac{du}{dt}(t) = \sum_{m=-\infty}^{\infty} s_m \cdot \frac{d\rho}{dt}(t-m) \tag{10}$$

where

$$\frac{d\rho}{dt} = \begin{cases} -\left(\omega^2+\beta^2\right)\cdot\left(\cos\omega t + \dfrac{\beta}{\omega}\sin\omega t\right)\cdot 2^t, & t<0 \\ 0, & t\ge 0 \end{cases} \tag{11}$$

is the derivative of the basis function. We define the $n^{\text{th}}$ return as

$$d_n = \frac{\sqrt{2}}{\omega^2+\beta^2}\cdot\frac{du}{dt}(n-1/2) \tag{12}$$

where $n$ is an integer value. The multiplicative factor is arbitrary and included for convenience. Using equation (10) with

$$\frac{d\rho}{dt}(n-m-1/2) = \begin{cases} \dfrac{1}{\sqrt{2}}\left(\omega^2+\beta^2\right)\cdot 2^{n-m}, & n\le m \\ 0, & n>m \end{cases} \tag{13}$$

we find the $n^{\text{th}}$ return in equation (12) yields

$$d_n = \sum_{m=0}^{\infty} s_{m+n}\cdot 2^{-m} \tag{14}$$

Similarly we find

$$d_{n+1} = \sum_{m=0}^{\infty} s_{m+n+1}\cdot 2^{-m} \tag{15}$$

is the next return. Examining equations (14) and (15), we immediately see that successive returns satisfy a shift relationship on the message symbols.

By considering all possible symbol sequences, we recognize equations (14) and (15) as binary expansions covering the closed interval [–2,2]. Furthermore, we recognize

$$s_n = \operatorname{sgn}(d_n) \tag{16}$$

where sgn is the signum function (ignoring the degenerate singular point $d_n = 0$). Using equations (14) and (16), we can write the next return (15) as

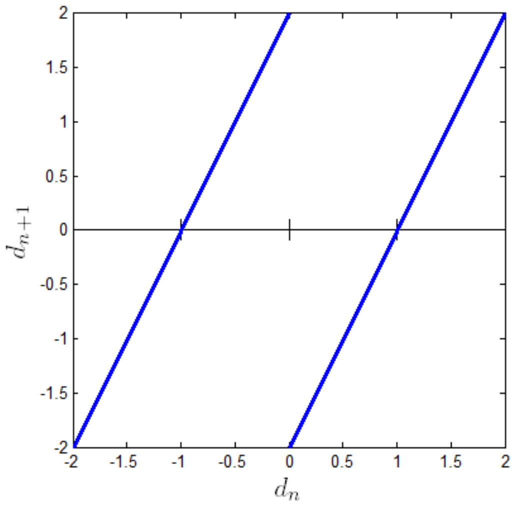


Figure 3. Return map derived from the time-derivative of the encoded waveform.

$$d_{n+1} = 2 \cdot \left[ d_n - \operatorname{sgn}\left(d_n\right) \right] \quad (17)$$

which provides an explicit return map for the waveform. This return map, which is closed on the interval [–2,2], is plotted in Figure 3. This return map is conjugate to a Bernoulli shift, which implies that orbits of the return map are chaotic. Since orbits of the map correspond to regular returns of the encoded waveform, the waveform is also chaotic (in the sense of Li-Yorke [2]), with positive Lyapunov exponent $\lambda = \ln 2$ . The waveform $u(t)$ is an example of a chaotic waveform constructed by linear superposition [3,4].

## 4. Differential Equation

It is desirable to find a means to generate the encoded waveform (3) with basis function (9). To this end, we note the waveform (3) is a particular solution of the differential equation

$$\frac{d^2u}{dt^2} - 2\beta \frac{du}{dt} + \left(\omega^2 + \beta^2\right)\left[ u - \sum_{m=-\infty}^{\infty} s_m \delta\left(t-m\right) \right] = 0 \quad (18)$$

which can be derived from equation (7) via time reversal and linear superposition. However, viewed as a dynamical system, this differential equation does not typically generate a solution in the form of equation (3). Negative damping in equation (18) implies a homogeneous solution with exponential growth; thus, the bounded waveform (3) is unstable.

To make a practical generator for the encoded waveform (3), we need to suppress the unbounded homogeneous solutions. Also, we find it desirable to incorporate the original message signal as defined in equation (1).

## 5. Transformed Equation

Moving toward a practical generator for the chaotic waveform, we first transform the differential equation to explicitly include the random signal in equation (1). To this end, we define a transformed waveform using the convolution

$$v(t) = \int_{-\infty}^{\infty} \phi(\tau) \cdot u(t-\tau)\, d\tau \quad (19)$$

where $\phi(\tau)$ is the square pulse defined in equation (2). Recognizing

$$\frac{dv}{dt}(t) = \int_{-\infty}^{\infty} \phi(\tau) \cdot \frac{du}{dt}(t-\tau)\, d\tau \quad (20)$$

and

$$\frac{d^2v}{dt^2}(t) = \int_{-\infty}^{\infty} \phi(\tau) \cdot \frac{d^2u}{dt^2}(t-\tau)\, d\tau \quad (21)$$

and using the sifting properties of the delta function, we formally apply the convolution in equation (19) to the differential equation (18). This operation yields the transformed differential equation

$$\frac{d^2v}{dt^2} - 2\beta \frac{dv}{dt} + \left(\omega^2 + \beta^2\right)\left(v - s\right) = 0 \quad (22)$$

where $s = s(t)$ is the random signal defined in equation (1).

Combining equations (3) and (19) allows us to write the transformed waveform as

$$v(t) = \sum_{m=-\infty}^{\infty} s_m \cdot P(t-m) \quad (23)$$

where

$$P(t) = \int_{-\infty}^{\infty} \phi(\tau) \cdot \rho(t-\tau)\, d\tau \quad (24)$$

is the transformed basis function. Direct evaluation of equation (24) yields

$$P(t) = \begin{cases} \frac{1}{2} \cdot \left( \cos\omega t - \frac{\beta}{\omega} \sin\omega t \right) \cdot 2^t, & t \le 0 \\ 1 - \frac{1}{2} \cdot \left( \cos\omega t - \frac{\beta}{\omega} \sin\omega t \right) \cdot 2^t, & 0 < t \le 1 \\ 0, & 1 < t \end{cases} \quad (25)$$

which is plotted with the digital basis function $\phi(t)$ in Figure 4. As such, the waveform (23) satisfies the transformed differential equation (22). It can be shown that the waveform (23) is also chaotic [4]. However, similar to equation (18), this differential equation admits an unbounded homogeneous solution which renders the particular solution (23) atypical.

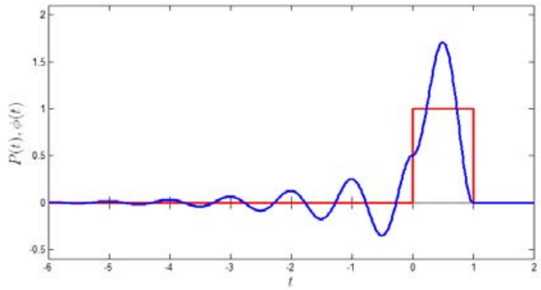


Figure 4. Transformed basis function $P(t)$ (blue) and corresponding digital basis function $\phi(t)$ (red).

## 6. Hybrid Oscillator

We have shown elsewhere that the waveform (23) with basis function (25) is an exact analytic solution to a chaotic hybrid oscillator [5], which is equivalent to an oscillator originally described by Saito and Fujita [6]. We write the hybrid oscillator as

$$\frac{d^2 v}{dt^2} - 2\beta \frac{dv}{dt} + \left(\omega^2 + \beta^2\right)(v - s) = 0 \qquad (26)$$

$$\frac{dv}{dt} = 0 \quad \Rightarrow \quad s = \operatorname{sgn}(v) \qquad (27)$$

where (27) defines a guard condition that sets the discrete state $s(t)$ to equal the sign of the continuous state $v(t)$ at each critical point in the waveform. This nonlinear dynamical system naturally generates a chaotic waveform in the form of equation (23). The hybrid oscillator is an example of exactly solvable chaos.

## 7. Transformed Matched Filter

Our analysis initiated with the desire for a correlation receiver with a simple matched filter. To complete our analysis, we require the matched filter for the transformed basis function $P(t)$. Using the operator notation, we equivalently seek the linear filter $\mathcal{L}$ such that

$$\mathcal{L} \circ \delta(t) = P(-t) \qquad (28)$$

where $\delta(t)$ is the impulse delta function and $P(t)$ is the basis function in equation (25). We elect to write the filter as the composition $\mathcal{L} = \mathcal{M}_2 \circ \mathcal{M}_1$, where $\mathcal{M}_1$ is the matched filter (4) used to define the basis function $\rho(t)$. Using equation (6), we then find

$$\mathcal{M}_2 \circ \bar{\rho}(t) = \bar{P}(t) \qquad (29)$$

where $\bar{P}(t) = P(-t)$ is the time reverse of the transformed basis function. From equation (24), we find

$$\bar{P}(t) = \int_{-\infty}^{\infty} \phi(-\tau) \cdot \bar{\rho}(t - \tau)\, d\tau \qquad (30)$$

which reveals the linear operator $\mathcal{M}_2$. Thus we define the filter

$$y(t) = \mathcal{M}_2 \circ x(t) \qquad (31)$$

as

$$y(t) = \int_{t}^{t+1} x(\tau)\, d\tau \qquad (32)$$

where $x(t)$ is its input and $y(t)$ is its output. Combining equation (4) with the differential version of the filter (32), we write the composite matched filter $\mathcal{L}$ as

$$\frac{d^2 x}{dt^2} + 2\beta \frac{dx}{dt} + \left(\omega^2 + \beta^2\right)(x - v) = 0 \qquad (33)$$

$$\frac{dy}{dt} = x(t+1) - x(t) \qquad (34)$$

where $v(t)$ is the input, $x(t)$ is an intermediate state, and $y(t)$ is the output [5]. That is, the differential equations (33) and (34) define an explicit matched filter for the transformed basis function (25).

## 8. Conclusions

We have shown that, under certain practical constraints, the optimal communication waveform is chaotic. The efficacy of this waveform was previously confirmed for communication [5] and, more recently, for detection and ranging [7].

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