IEICE Proceeding Series

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Vol. 1 pp. 215-218
Publication Date: 2014/03/17
Online ISSN: 2188-5079

2012 International Symposium on Nonlinear Theory and its Applications
NOLTA2012, Palma, Majorca, Spain, October 22-26, 2012

Continuous Global Minimization Method Based on Special Mathematical Structure of Objective Functions and Adjacent Local Minima Search

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Abstract—We introduce mathematical structure of local minima, and propose a concept of *adjacent local minima* in univariate multimodal functions. We rewrite the our previous mathematical structure using the concept. We propose an method for finding the global minimum of a multivariate function whose univariate function on line search is almost lower unimodal sequence. We show using a numerical example that the method effectively finds the global minimum with only a few function evaluations.

1. Introduction

Many methods have been proposed for solving a global optimization (minimization) problem of a real valued function f of n-variables with bounded constraints.

Those methods can be mainly classified as deterministic approach and stochastic approach. The deterministic approach is based on a branch and bound approach. On the other hand, some methods using a stochastic approach are based on random sampling and local search, then many recently proposed methods (SA, GA, DE, etc.) of this approach can be included in a heuristic framework [1].

However, searching spaces or sample size of those approaches exponentially increase with increase in the number of dimensions n in the problem. This phenomenon, known as the "curse of dimensionality", led to the abandonment of those search methods in favour of ones using some *a priori* knowledge or *priori* structure of the function.

In this paper, we consider a special structure of a univariate function f on an interval $[a, b]$ such that the sequence of local minimal function values is lower unimodal. For the type of functions has been already described in our paper [3]. However, functions treated in our previous paper are more restricted for multivariate functions using the structure, that is, conventional method is restricted to local minima run parallel to each coordinate.

The purpose of this paper is to study the mathematical structure of local minima (maxima) and of univariate multimodal functions, and to study a non-separable multivariate function whose local minimal values of the univariate function on line segment in n-dimensional search space is a almost unimodal sequence. We propose a effective algorithm for finding a global minimum of the type of functions.

The remainder of the paper is organized as follows. A problem and mathematical structure of univariate problem are given in sect. 2. In sect. 3, algorithms for univariate functions are presented. In sect. 4, an algorithm for globally minimizing multivariate functions iterative using line search is presented. The results of a numerical experiment and concluding remarks are shown in sect. 5 and 6.

2. Preliminary

2.1. Problem and mathematical structures of local minima(maxima)

In this section and the next section, we consider a univariate minimization problem (P1):

$$\text{min.}\ \ f(x), \quad x \in D \equiv [a, b] \subset \mathbb{R}, \tag{P1}$$

Suppose f is a twice continuous function, and all local minima of f in $[a, b]$ are isolated. These minima are denoted by $a < x_*^1 < x_*^2 < \cdots < x_*^M < b$, and these function values are denoted by $f_i^* \equiv f(x_*^i)$ $(i = 1, 2, \ldots, M)$.

Definition 1 In problem (P1), the function f has a *strictly lower unimodal sequence* (hereafter called a *unimodal sequence*) in the sequence of local minimal function values, if there exists $k \in [2, M-1]$ such that

$$\begin{cases} x_*^1 < x_*^2 < \cdots < x_*^{k-1} < x_*^k < x_*^{k+1} < \cdots < x_*^M, \\ f_*^1 > f_*^2 > \cdots > f_*^{k-1} > f_*^k < f_*^{k+1} < \cdots < f_*^M, \end{cases} \tag{1}$$

is called *unimodal local minimal values function*.

Definition 2 In eq.(1), if the following equation:

$$\forall x_*^{i_1},\ x_*^{i_2}\ (1 \le i_1 < i_2 \le M);\ \ i_2 - i_1 = 1, \tag{2}$$

holds, then $x_*^{i_1}$, $x_*^{i_2}$ are called *adjacent local minima*.

The following theorem holds in the problem (P1)[4].

Theorem 1 A necessary and sufficient condition such that local minima of a function are all isolated on compact set is that number of local minima is finite. Moreover, if a univariate function is continuous the following property holds[3].

Theorem 2 If f is continuous and its all local minima are all isolated on $[a, b]$, then there exists strictly monotonically decreasing(increasing) region on the left(right) side of each local minimum.

From the theorem, the following properties can be easily shown[4].

Property 1 If a function f is continuous on an interval $[a, b]$ and its all local minimum are all isolated, then the

following three properties holds.
1) All local maxima on $[a, b]$ are all isolated.
2) If local minima and maxima rearrange in ascent order, local minima and local maxima alternately line up.
3) There exists the strictly monotonic decreasing (increasing) region on left (right) side of a local minimum (local maximum).

From the above study and assumptions of the problem, the following equation holds by repeatedly such a arrange of local minima and maxima,

$$a \le \overline{x}_*^0 < x_*^1 < \overline{x}_*^1 < x_*^2 < \cdots < \overline{x}_*^{M-1} < x_*^M < \overline{x}_*^M \le b \qquad (3)$$

2.2. Mathematical structure of multimodal functions

A *unimodal region* $R_u(x_*^i)$ of a local minimum x_*^i of a function f is defined as the maximum region such that the function f is unimodal[3]. By using **Property 1**, the previous definition is more simply formulated as follows.

Definition 3 The unimodal region is defined as follows

$$R_u(x_*^i) \equiv [\overline{x}_*^{i-1},\ \overline{x}_*^i]. \qquad (4)$$

Definition 4 In the unimodal region $R_u(x_*^i)$ at x_*^i, its width $w(x_*^i)$ and radius $r(x_*^i)$ are formulated as follows.

$$w(x_*^i) = \overline{x}_*^i - \overline{x}_*^{i-1}, \quad r(x_*^i) = \min\{x_*^i - \overline{x}_*^{i-1}, \overline{x}_*^i - x_*^i\}. \qquad (5)$$

From the definition, the maximum and minimum widths of unimodal regions can be expressed as follows.

$$\overline{w} = \max_{1\le i\le M} w(x_*^i), \qquad \underline{w} = \min_{1\le i\le M} w(x_*^i). \qquad (6)$$

We show definitions on depths of a unimodal region as mathematical structure of multimodal functions.

Definition 5 *Deeper depth* $\overline{d}(x_*^i)$ and *Shallower depth* $\underline{d}(x_*^i)$ of the unimodal region $R_u(x_*^i)$ are defined as follows.

$$\begin{cases} \overline{d}(x_*^i) \equiv \max\{f(\overline{x}_*^{i-1}) - f(x_*^i),\ f(\overline{x}_*) - f(x_*^i)\}, \\ \underline{d}(x_*^i) \equiv \min\{f(\overline{x}_*^{i-1}) - f(x_*^i),\ f(\overline{x}_*) - f(x_*^i)\}. \end{cases} \qquad (7)$$

3. Algorithm for Univariate Functions

The idea of our previously proposed algorithm [3] is to use a two-stage minimizer, 1) a *large-step minimizer* and 2) a *small-step local minimizer*, in each iteration. An outline of the two steps is as follows.

1) The *large-step minimizer* generates new points $x^0, x^1, \ldots$ such that any two points are included in different unimodal regions, that is

$$\begin{aligned} &\forall x^m \ne \forall x^n \text{ and } x^m \in R_u(x_*^i),\ x^n \in R_u(x_*^j) \\ &\quad \Longrightarrow \quad \mathrm{int}\left(R_u(x_*^i)\right) \cap \mathrm{int}\left(R_u(x_*^j)\right) = \emptyset, \end{aligned} \qquad (8)$$

where int(·) is an interior of a set.

2) The *small-step local minimizer* finds a local minimum x_*^i in a unimodal region $R_u(x_*^i)$ from a starting point $x^k \in R_u(x_*^i)$ generated by the *large-step minimizer* :

$$\text{for } x^k \in R_u(x_*^i),\ \ x_*^{(k)} \leftarrow ML(x^k, \overline{\delta}) \ \Longrightarrow\ x_*^{(k)} = x_*^i, \qquad (9)$$

where $ML(x^k, \overline{\delta})$ is a procedure of the *small-step local minimizer* with a starting point x^k and small step $\overline{\delta}$.

By the above investigation, it is concluded that points $x^0, x^1, \ldots$ generated by the *large-step minimizer* converge to different local minima $x_*^{(0)}, x_*^{(1)}, \ldots$ by the *small-step local minimizer*, that is

$$\begin{aligned} &\forall x^m \ne \forall x^n, \quad x_*^{(m)} \leftarrow ML(x^m, \overline{\delta}), \quad x_*^{(n)} \leftarrow ML(x^n, \overline{\delta}) \\ &\quad \Longrightarrow x_*^{(m)} \ne x_*^{(n)}. \end{aligned} \qquad (10)$$

In order to satisfy equations (8), distance of two mutually different points in the sequence $x^0, x^1, \ldots$ generated by the *large-step minimizer* must greater than the maximum width of unimodal region $\overline{w}$, as follows:

$$\forall x^m \ne \forall x^n, \quad |x^m - x^n| > \overline{w}. \qquad (11)$$

From the definition (1) of a unimodal sequence, for three points $x_*^{(p)} > x_*^{(q)} > x_*^{(r)}$, the following condition of enclosing the global minimum x_{**} holds.

$$f(x_*^{(p)}) > f(x_*^{(q)}) < f(x_*^{(r)}) \ \Longrightarrow x_{**} \in (x_*^{(p)}, x_*^{(r)}) \qquad (12)$$

The outline of the previous algorithm is as follows.

S1p. Bracketing a minimum x_{**} by an interval $[x_*^{(p)}, x_*^{(r)}]$ such that $f(x_*^{(p)}) > f(x_*^{(q)}) < f(x_*^{(r)})$, $x_*^{(p)} < x_*^{(q)} < x_*^{(r)}$.

S2p. Reducing the interval $[x_*^{(p)}, x_*^{(r)}]$ such that $x_{**} \in [x_*^{(p)}, x_*^{(r)}]$ until the following stop condition holds.
$f_*^{(p)} > f_*^{(q)} < f_*^{(r)}, \quad x_*^{(r)} - x_*^{(p)} \approx 2\overline{w}$.

S3p. Apply a one-dimensional global minimization algorithm for the interval $[x_*^{(p)}, x_*^{(r)}]$.

From the above investigations, the specification of the algorithm *MGuf* that finds the global minimum x_{**} and its function value f_{**} of a function $f(x)$ in a searching region $D = [a, b]$ for a given initial point x^0, its function value f^0, an initial step size Δ, an upper limit of step size $\overline{\delta}$, maximum width $\overline{w}$ and the minimum width $\underline{w}$ of unimodal regions and a tolerance ε as follows:

$$(f_{**}, x_{**}) \leftarrow MGuf(\ f,\ D,\ f^0,\ x^0,\ \Delta,\ \overline{\delta}, \overline{w},\ \underline{w},\ \varepsilon\ ).$$

4. Algorithm and Results for Multivariate Functions

4.1. Outline of the previous algorithm

We consider the following problem (Pn):

$$\text{min.}\ \ f(\mathbf{x}) \quad \mathbf{x} \in D^n \equiv \prod_{j=1,\ldots,n} D_j \equiv \prod_{j=1,\ldots,n} [a_j, b_j]. \qquad \text{(Pn)}$$

However, it cannot be formulated for multivariate function $f(\mathbf{x})$ like eq.(1). To overcome this problem, we consider a problem with a univariate function ϕ as follows:

$$\alpha^{(k)} = \underset{\alpha}{\mathrm{argmin}}\{\phi(\alpha) \equiv f(\mathbf{x}^{(k)} + \alpha \mathbf{d}^{(k)})\}, \qquad \text{(PnL)}$$

where $\mathbf{x}^{(k)}$ is the starting point and $\mathbf{d}^{(k)}$ is the searching direction. Moreover, we assume that the function ϕ almost satisfies equation (1), and the function ϕ is globally minimized by applying the previous globally minimization algorithm. Such a minimization step is called a *line search*, and the step is usually used at iteration k in minimization methods with updating of the new point $\mathbf{x}^{(k+1)}$:

$$\mathbf{x}^{(k+1)} \leftarrow \mathbf{x}^{(k)} + \alpha^{(k)} \mathbf{d}^{(k)}, \quad (k = 0, 1, 2, \ldots). \qquad \text{(a)}$$

Generally, direction $\boldsymbol{d}^{(k)}$ is usually determined by gradient $\nabla f(\boldsymbol{x}^{(k)})$ of function f. However, if the line search step is performed along this direction, the obtained point $\boldsymbol{x}^{(k+1)}$ almost falls into a local minimum. In order to avoid this problem, the i-th element $d_i^{(k)}$ of $\boldsymbol{d}^{(k)}$ is determined by two adjacent local minima α_*^1, α_*^2 from $\boldsymbol{x}^{(k)}$ along i-th coordinate direction $\boldsymbol{e}_i$ and both function values as follows:

$$d_i^{(k)} = -\frac{\phi_i(\alpha_*^1) - \phi_i(\alpha_*^2)}{\alpha_*^1 - \alpha_*^2}, \quad \phi_i(\alpha) \equiv f(\boldsymbol{x}^{(k)} + \alpha \boldsymbol{e}_i). \tag{13}$$

At last step, a local minimization method $MLnf(\cdot)$ is applied to problem (Pn) for finding a more accurate solution.

We show an algorithm for finding the global minimum $\boldsymbol{x}_{**}$ and its function value f_{**} from a point $\boldsymbol{x}^{(0)}$ for function f with maximum width $\overline{w}$ and minimum width $\underline{w}$ as follows.

$(f_{**}, \boldsymbol{x}_{**}) \leftarrow MGnf1(f, D^n, f^{(0)}, \boldsymbol{x}^{(0)}, \Delta, \overline{\delta}, \overline{w}, \underline{w}, \varepsilon, \varepsilon_g)$
1) Initialize: $k \leftarrow 0$; $f^{(0)} \leftarrow f(\boldsymbol{x}^{(0)})$;
repeat
2) Compute $\boldsymbol{d}^{(k)}$ by eq.(13) and set: $\phi(\alpha) \equiv f(\boldsymbol{x}^{(k)} + \alpha \boldsymbol{d}^{(k)})$;
3) Apply global line search:
$(f^{(k+1)}, \alpha^{(k)}) \leftarrow MGuf(\phi, D, f^{(k)}, x^{(k)}, \Delta, \overline{\delta}, \overline{w}, \underline{w}, \varepsilon)$;
4) Update for the next iteration:
$\boldsymbol{x}^{(k+1)} \leftarrow \boldsymbol{x}^{(k)} + \alpha^{(k)} \boldsymbol{d}^{(k)}$; $f^{(k+1)} \leftarrow f(\boldsymbol{x}^{(k+1)})$; $k \leftarrow k+1$;
until $\|\alpha^{(k)} \boldsymbol{d}^{(k)}\| < \varepsilon_g$.
5) Apply local search $MLnf$ to the point $\boldsymbol{x}^{(k)}$ and its function value: $f^{(k)}$: $(f_{**}, \boldsymbol{x}_{**}) \leftarrow MLnf(f^{(k)}, \boldsymbol{x}^{(k)}, \varepsilon)$.

4.2. Outline of a new algorithm

In eq.13, if local maxima exists parallel to each coordinate, $\boldsymbol{d}_i^{(k)}$ will become a good approximation of i-th element of steepest descent direction on lower envelope of function f. However it cannot be assumed that local maxima exists parallel to each coordinate, the difference found by several local minima, it is possible to give a good approximation of the above i-th element of direction.

Let simplex points that consists of $(n+1)$-approximated local minimum(a.l.m.) be $\{\tilde{\boldsymbol{x}}_*^j\}$ $(j = 0, \ldots, n)$. In case where, $n \times n$ matrix V_*^S of simplex directions from first point $\tilde{\boldsymbol{x}}_*^0$ to n number of point $\tilde{\boldsymbol{x}}_*^1, \tilde{\boldsymbol{x}}_*^2, \ldots, \tilde{\boldsymbol{x}}_*^n$ is denoted by

$$V_*^S = (\tilde{\boldsymbol{x}}_*^1 - \tilde{\boldsymbol{x}}_*^0,\ \tilde{\boldsymbol{x}}_*^2 - \tilde{\boldsymbol{x}}_*^0,\ \ldots,\ \tilde{\boldsymbol{x}}_*^n - \tilde{\boldsymbol{x}}_*^0). \tag{14}$$

Similarly, n-differences between the function value at $\tilde{\boldsymbol{x}}_*^0$ and function values at the other n-points is denoted by

$$\Delta_*^f = (\tilde{\boldsymbol{f}}_*^1 - \tilde{\boldsymbol{f}}_*^0,\ \tilde{\boldsymbol{f}}_*^2 - \tilde{\boldsymbol{f}}_*^0, \ldots, \tilde{\boldsymbol{f}}_*^n - \tilde{\boldsymbol{f}}_*^0)^T,\ \tilde{\boldsymbol{f}}_*^i \equiv f(\tilde{\boldsymbol{x}}_*^i)\ (i = 1, \ldots, n).$$

Using eq.(14) and eq.(15), simplex gradient $\widetilde{\nabla}_* f$ and searching direction $\boldsymbol{d}$ is determined as follows[5].

$$\widetilde{\nabla}_* f = \left((V_*^S)^T\right)^{-1} \Delta_*^f, \quad \boldsymbol{d} = -\widetilde{\nabla}_* f. \tag{15}$$

We show an algorithm for finding the global minimum $\boldsymbol{x}_{**}$ and its function value f_{**} from a point $\boldsymbol{x}^{(0)}$ for function f with tolerance ε_g and ε as follows.

$(f_{**}, \boldsymbol{x}_{**}) \leftarrow MGnf2(f, D^n, f^{(0)}, \boldsymbol{x}^{(0)}, \Delta, \overline{\delta}, \varepsilon, \varepsilon_g)$
1) Initialize: $k \leftarrow 0$; $f^{(0)} \leftarrow f(\boldsymbol{x}^{(0)})$;
Find first l.m. $\tilde{\boldsymbol{x}}_*^0$ and its function value $\tilde{\boldsymbol{x}}_*^0$ by local search: $(\tilde{\boldsymbol{f}}_*^0, \tilde{\boldsymbol{x}}_*^0) \leftarrow MLnf(f^{(0)}, \boldsymbol{x}^{(0)}, \varepsilon)$.
repeat
2) Apply a.l.m.-search from $\tilde{\boldsymbol{x}}_*^{0(k)}$ along i-coordinate directions $\boldsymbol{e}_i$ $(i = 1, 2, \ldots, n)$, find adjacent local minima $\tilde{\boldsymbol{x}}_{*L}^i$ $(i = 1, .., M_1)$ to $\tilde{\boldsymbol{x}}_*^0$.
3) Find n-a.l.m. and its function values by $\overline{k}^i$-step local search:
$(\tilde{\boldsymbol{f}}_*^i, \tilde{\boldsymbol{x}}_*^i) \leftarrow MLnf^r(\tilde{f}_{*L}^i, \tilde{\boldsymbol{x}}_{*L}^i, \overline{k}^i, \varepsilon)$, $(i = 1, 2, \ldots, n)$.
4) Compute $\boldsymbol{d}^{(k)}$ by eq.(15) and set:
$\boldsymbol{x}^{(k)} = 1/(n+1) \sum_{i=0}^n \tilde{\boldsymbol{x}}_*^i$; $f^{(k)} = f(\boldsymbol{x}^{(k)})$;
$\phi(\alpha) \equiv f(\boldsymbol{x}^{(k)} + \alpha \boldsymbol{d}^{(k)})$;
5) Apply global line search:
$(f^{(k+1)}, \alpha^{(k)}) \leftarrow MGuf1(\phi, D, f^{(k)}, x^{(k)}, \Delta, \overline{\delta}, \varepsilon)$;
6) Update for the next iteration:
$\boldsymbol{x}^{(k+1)} \leftarrow \boldsymbol{x}^{(k)} + \alpha^{(k)} \boldsymbol{d}^{(k)}$; $f^{(k+1)} \leftarrow f(\boldsymbol{x}^{(k+1)})$; $k \leftarrow k+1$;
until $\|\alpha^{(k)} \boldsymbol{d}^{(k)}\| < \varepsilon_g$.
7) Apply local search $MLnf$ to the point $\boldsymbol{x}^{(k)}$ and its function value $f^{(k)}$: $(f_{**}, \boldsymbol{x}_{**}) \leftarrow MLnf(f^{(k)}, \boldsymbol{x}^{(k)}, \varepsilon)$.

Where, $MLnf^r(\cdot)$ is $\overline{k}^i$-step local search whose inner limit of iteration is restricted to $\overline{k}^i$ at step **3)**.

5. Numerical Experiments

Conditions of this experiments and input parameters of algorithms are as follows.

- Our algorithm is performed 100 times per problem by randomly generating initial points $\boldsymbol{x}^{(0)}$ in D^n.
- After the above 100 times trials per each problem, mean of number of function evaluations $\overline{N}_e$ and mean of obtained minimal function values $\overline{f}_{**}$ are calculated.
- For problems with periodic term on a objective function, $\overline{w}$ and $\underline{w}$ are set per problem. For problems with a unimodal objective function, set $\overline{w} = \underline{w} = \min_{1 \le i \le n} b_i - a_i$. For the other problem, these bounds set 0 .
- The other input parameter is set $\Delta^{(0)} = 3\overline{w}$, $\overline{\delta} = 0.2\underline{w}$, $\varepsilon_g = 0.4\underline{w}$ and $\varepsilon = 1.0 \times 10^{-5}$.
- Main program selects $MGnf1(\cdot)$ in case where $\overline{w} > 0$, and selects $MGnf2(\cdot)$ in case where $\overline{w} = 0$.

The outline of other methods for comparison with our method are shown in table 1.

A problem of n-variables with M-local minima is denoted by $Prb._{n,M}$, where Prb. is an abbreviated name.

Table 1: Outline of methods for our comparison

abbrev. name	method	author(year)
DEPD	Differentical Evolution using Pre-calculated Differentials	Ali and Törn[1](2002)
SCGA	Simplex Coding Genetic Algorithm(GA)	Hedar and Fukushima[2](2003)
GPSA	Combined Algorithm of GA and digital Pattern Search	Kim, et al.[6](2009)

5.1. Problems with periodic term

- Griewank Problem($n = 10$, several thousands of local minima; $G_{10,\widetilde{5000}}$)

$$\begin{cases} \text{min.} & f_G(\boldsymbol{x}) = 1 + \sum_{i=1}^{10} x_i^2/4000 - \prod_{i=1}^{10} \cos(x_i/\sqrt{i}), \\ \text{sub.} & \boldsymbol{x} \in D^{10} = [-600, 600]^{10}. \end{cases}$$

The minimum of the problem is $\boldsymbol{x}_{**} = (0, \ldots, 0)$, and its minimal function value is $f_{**} = 0$.

Input parameters are set to $\underline{w} = 2\pi$, $\overline{w} = 2\sqrt{10}\pi$ based on periods of $\prod_{i=1}^{10} \cos(x_i/\sqrt{i})$, respectively. In 100 trials for the problem, the mean of number of function evaluations is $\overline{N}_e = 562.6$, and the mean of obtained minimal function value is $\overline{f}_{**} = 1.27 \times 10^{-10}$.

- Rastrigin problem($n = 10$, $M = 11^{10}$; $R_{10,11^{10}}$)

$$\begin{cases} \text{min.} & f_R(\boldsymbol{x}) = 10n + \sum_{i=1}^{10} \left(x_i^2 - 10\cos(2\pi x_i)\right) \\ \text{sub.} & \boldsymbol{x} \in [-5.12, 5.12]^{10}. \end{cases}$$

The minimum of the problem is $\boldsymbol{x}_{**} = (0, 0, \ldots, 0)$, and its minimal function value is $f_{**} = 0$.

Input parameters are set to $\overline{w} = \underline{w} = 1$ based on period of term $-10\cos(2\pi x_i)$ of f.

In 100 trials for the problem, the mean of number of function evaluation is $\overline{N}_e = 381.6$, and the mean of obtained minimal function value is $\overline{f}_{**} = 4.26 \times 10^{-14}$.

5.2. Problem with unimodal objective function

- Zakharov problem($n = \{2, 5, 10, 20\}$, $M = 1$; $Z_{\{2,5,10,20\},1}$)

$$\begin{cases} \text{min.} & f_Z(\boldsymbol{x}) = \left(\sum_{i=1}^{n} 0.5 i x_i\right)^4, \\ \text{sub.} & \boldsymbol{x} \in D^n = [-5, 10]^n, \quad n = 2, 5, 10, 20. \end{cases}$$

The minimum of the problem is $\boldsymbol{x}_{**} = (0, 0, \ldots, 0)$, and its minimal function value is $f_{**} = 0$.

In 100 trials for the problem, the mean of number of function evaluations are $\overline{N}_e = \{89.9(n = 2),\ 161.2(n = 5),\ 255.2(n = 10),\ 460.1(n = 20)\}$, and the mean of obtained minimal function value are $\overline{f}_{**} = \{0.313(n = 2),\ 2.02(n = 5),\ 1.84(n = 10),\ 1.67(n = 20)\} \times 10^{-12}$.

5.3. Problems with general multimodal function

- Shekel problems($n = 4$, $M = \{5, 7, 10\}$; $S_{4,\{5,7,10\}}$)

$$\begin{cases} \text{min.} & f_S(\boldsymbol{x}) = \sum_{j=1}^{M} \left(\sum_{i=1}^{4} (x_i - C_{ij})^2 + \beta_j\right)^{-1}, \\ \text{sub.} & \boldsymbol{x} \in D^4 = [0, 10]^4, \quad M = 5, 7, 10. \end{cases}$$

5.4. Comparison of results

Comparisons of our method and other methods is shown in table 2. In comparisons of results on problems with periodic term, showed that our method finds the global minimum of 7.5%–0.38% in the number of function evaluations of DEPD method. In comparisons on problems with unimodal function and general multimodal functions, showed that the method effectively finds the global minimum.

Table 2: Comparison our method and other methods for Problem(Prb.$_{n,M}$) of n variables and M-local minima

Prb.$_{n,M}$	**ALMS**	GPSA	SCGA	DEPD
$G_{10,5\tilde{0}00}$	563/100	N.A.	N.A.	7,556/100
$R_{10,11^{10}}$	382/100	N.A.	N.A.	101,417/100
$Z_{2,1}$	90/100	491/100	170/100	N.A.
$Z_{5,1}$	161/100	1,138/100	998/100	N.A.
$Z_{10,1}$	255/100	7,576/100	1,829/100	N.A.
$Z_{20,1}$	460/100	102,908/100	N.A.	N.A.
$S_{4,5}$	482/100	456/34	1,086/79	4,351/100
$S_{4,7}$	485/100	464/21	1,087/87	3,614/100
$S_{4,10}$	515/100	484/24	1,068/84	3,489/100

(*) Each element denote $N_e/S.R.$(S.R.:success % rate).
(**) **ALMS**: Our method(Adjacent Local Minima Search).

6. Conclusions

We have mainly proposed *adjacent of local minima* and *depth of a local minima* as mathematical structure of a univariate function with isolated minima on an interval constraint. We introduce the already proposed algorithm for special univariate functions whose local minimal function values have unimodal sequence. Moreover, we investigated two types of multivariate function similar to the above special type of function, and propose two algorithms based on adjacent local minima search. The results of a numerical example showed that the algorithm effectively finds the global minimum with only a few function evaluations.

References

[1] Dzemyda, G. , Šaltenis, V. and Žilinskas, A. : "Stochastic and Global Optimization," Kluwer Acad. Publ., Dordrecht, 2002.

[2] A. Hedar and M. Fukushima: "Minimizing multimodal functions by simplex coding genetic algorithm," Optimization Methods and Software, vol.18, 265–282, 2003.

[3] Kanemitsu, H., et al.: An Algorithm for Globally Minimizing a Function with Local Minimal Values whose Sequence is Almost Lower Unimodal, Proc. of NOLTA'2009, 580–583, 2009.

[4] Kanemitsu, H.: "Studies on Continuous Global Optimization Methods using Mathematical Structure of Problems," Doctoral dissertation, Hokkaido University, 2011(In Japanese).

[5] C.T. Kelly, "Iterative Methods for Optimization,"SIAM, 1999.

[6] N.-G. Kim, Y. Park, J.-W. Kim, E. Kim and S.W. Kim, "Digital Pattern Search and Its Hybridization with Genetic Algorithms for Bound Constrained Global Optimization," IEICE-EA, vol.E92-A, 481–492, 2009.