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A Visualization Method for the Stretch-and-fold Mechanism in Chaotic Dynamics

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Abstract—Methods of visualizing the stretch-and-fold mechanism in chaotic dynamical systems helps us determine whether an observed time-series is chaotic or not, and let us analyze the dynamics more precisely. We have already proposed a semi-automatic method for visualizing the stretch-and-fold mechanism, which can be applied to chaotic dynamical systems even if the structure of the chaotic dynamical systems is unknown. In this paper, we extend this method using the entropy and we show that the proposed method can detect the stretch-and-fold mechanism more clearly. We also show that the proposed method can perform accurately as the conventional method, and can be applied to a high-dimensional chaotic dynamical system and a time-series by numerical simulations.

1. Introduction

In our daily life, we can often observe various kinds of complex phenomena, which might be generated from deterministic nonlinear, possibly chaotic dynamical systems. Therefore, it is important to analyze characteristic properties of the nonlinear dynamics to realize novel engineering applications. Among several essential characteristics of the nonlinear dynamics, stretch-and-fold mechanism[1] is one of the most important ones to produce chaotic dynamics. Several methods for analyzing the stretch-and-fold mechanism have been proposed. The Poincaré section is a basic and important analysis method to detect and evaluate the stretch-and-fold mechanism. Although the Poincaré section often works well to detect the stretch-and-fold mechanism, we have to appropriately assign the transversal sections on chaotic attractors when we use the Poincaré sections. To solve this issue and realize a semi-automatic assignment, we have already proposed a method[2, a, b] based on the Lorenz plot[3].

In this paper, by using the entropy, we extend the method based on the Lorenz plot to visualize the stretch-and-fold mechanism embedded in the chaotic dynamics more clearly, and to make it possible to analyze quantitatively. We show that these methods can detect the stretch-and-fold mechanism clearly enough as the Poincaré section method. We also examine the performance of the method when it is applied to a high-dimensional chaotic dynamical system and a chaotic time-series. Through these examination, we show that the method can be applied to real data which are often obtained as a time-series and are embedded in a high-dimensional state space.

2. Methods detecting the stretch-and-fold mechanism

The Poincaré section detects the stretch-and-fold mechanism embedded in a chaotic attractor by assigning transversal sections manually on the chaotic attractor. On the Poincaré section, intersection of trajectories of the chaotic attractor is presented by points on the section. Based on the Poincaré section, the Lorenz plot section[2, a, b] identifies a Poincaré map on a curved surface semi-automatically by introducing the Lorenz plot[3]. In the method, first, local maxima of a time-series $x(t)$ produced or observed from a nonlinear dynamical system are identified as $X(n) \equiv x(t_0(n))$, where $t_0(n)$ is the occurrence time of the nth local maximum. Then, the time interval between $t_0(n)$ and $t_0(n+1)$ is equally divided by $k-1$ points, $t_i(n)$ $(i = 1, 2, \cdots, k-1)$. Finally, the ith divided points are plotted on the ith surface (Fig.1). Figure 2 shows the Lorenz plot

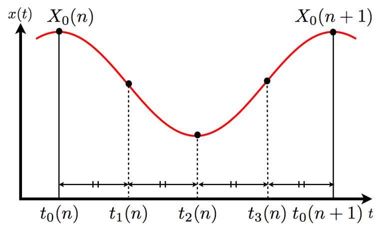


Figure 1: The method to set the points on the Lorenz plot section.

section on the Rössler attractor derived from the Rössler equations[4]:

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + 0.23y, \\ \dot{z} = 0.34 - 4.5z + xz. \end{cases} \quad (1)$$

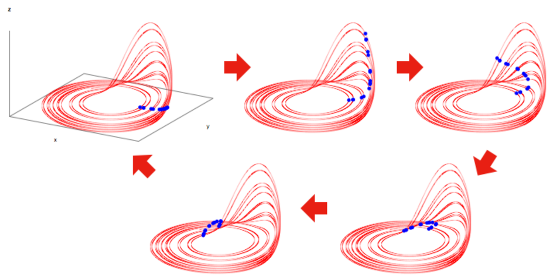


Figure 2: Applying the Lorenz plot section method to a chaotic Rössler attractor. Blue points exist on the curved Lorenz plot sections.

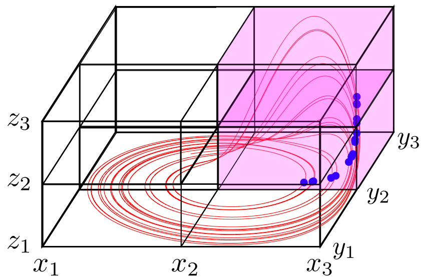


Figure 3: The method to calculate the entropy when $l = 3$. The highlighted boxes (in this example, two boxes) include points of the section.

From Fig.2, we can detect the stretch-and-fold mechanism clearly by moving the section and observe points plotted on the sections getting stretched and folded. However, it may be difficult to take the same way if we have to detect the stretch-and-fold mechanism in more complex systems or when the dimension of a system is higher. Therefore, we use the entropy to evaluate the stretch-and-fold mechanism in a more specific way.

We assume that the entropy of distributed points on the section will become higher when a set of points is stretched and lower when it is folded. Based on this idea, we evaluated how the entropy of points on the sections will change by moving the phases of the sections. First, we defined a rectangular box which exactly bounds the attractor by using the maximum and minimum values ($x_{\max}, x_{\min}, y_{\max}, y_{\min}, z_{\max}$ and $z_{\min}$) of three variables (x, y and z) of possible chaotic attractors (Fig.3). Then, in this rectangular box, we defined $(l-1)^3$ infinitesimal rectangular boxes by dividing the ranges ($x_{\min} \le x \le x_{\max}$, $y_{\min} \le y \le y_{\max}$ and $z_{\min} \le z \le z_{\max}$) of each variable equally into l pieces. Thus, each infinitesimal rectangular box is bounded by $[x_i, x_{i+1}]$, $[y_j, y_{j+1}]$ and $[z_k, z_{k+1}]$$(i, j, k = 1, 2, \ldots, l-1)$. By using these infinitesimal rectangular boxes, we defined the entropy

$$H = -\sum_{m=1}^{(l-1)^3} (P_m/N)\log(P_m/N), \qquad (2)$$

where P_m denotes the number of points included in the mth infinitesimal rectangular box and N denotes the number of all points. Next, to remove arbitrariness of placements of the boxes, we calculated an average value of the entropy H_{ave} at all possible placement by shifting the boundaries ($x_{\max}, x_{\min}, y_{\max}, y_{\min}, z_{\max}$, and $z_{\min}$).

3. Embedding

When we analyze properties of an observed time-series, we often reconstruct it in an m-dimensional state space by using a time-delay coordinate. In this method, when the time-delay and the dimension m are set appropriately, diffeomorphism of a dynamical structure of the embedded time-series and the unknown original dynamical system of the time-series is guaranteed by the Takens embedding theorem[5]. To reconstruct the state space, an observed time series $x(1), x(2), x(3), \cdots x(N)$ is transformed by using the following equation:

$$u(t) = (x(t), x(t+\tau), \ldots, x(t+(m-1)\tau)) \qquad (3)$$

where, τ is the time-delay and m is a dimension of a reconstructed attractor. In this paper, we set τ to the first time when the mutual information takes the minimum value, so that the coordinate values of $u(t)$ are less correlated[6], and $m = 3$ considering the calculation time.

4. Results and Discussions

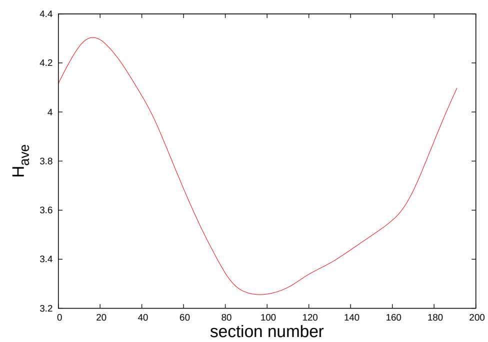


Figure 4: Entropy change of the Poincaré section of the Rössler attractor. The average values H_{ave} of the entropy H of the sections are plotted with 1,000 trials. The number of sections is 192 and $l = 50$.

First, we show how the entropy H_{ave} of Poincaré section of the Rössler attractor changes in Fig.4. We will use the Poincaré section as a standard method, because the Poincaré section method can detect the stretch-and-fold mechanism quite clearly. From Fig.4, first, the entropy gradually increases, and when it gets to the peak, it gradually decreases. This means that the points plotted on the sections are getting stretched and folded, which indicates that the stretch-and-fold mechanism is detected.

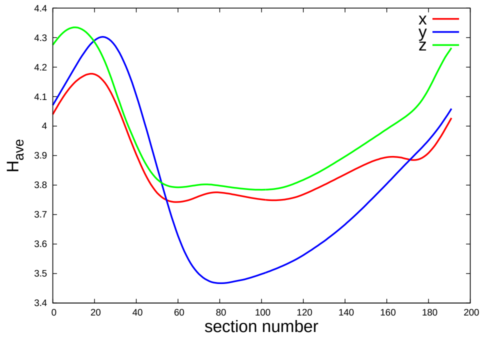


Figure 5: Entropy change of the Lorenz plot section of the Rössler attractor. The Lorenz plot section method is applied to all variables of the Rössler equations described in Eq.(1): x(red), y(blue) and z(green). The average values $H_{\rm ave}$ of the entropy H of the sections are plotted with conducting $1,000$ trials. The number of sections is 192 and $l = 50$.

Table 1: Correlation coefficients of entropy of the Poincaré section method and the Lorenz plot section method applied to the same Rössler attractor.

variables	x	y	z
correlation coefficients	0.687	0.904	0.837

Figure 5 shows how the entropy $H_{\rm ave}$ changes when the Lorenz plot section method is applied to each variable of the Rössler equations. We can see the entropy gradually getting higher, and then lower, which shows the same characteristic as the entropy change of the Poincaré section. Next, to evaluate the performance of the Lorenz plot section method more precisely, we examined how much the entropy change of the Lorenz plot section method differ from that of the Poincaré section method applied to the same attractor by using correlation coefficients. The results are summarized in Table 1. From Table 1, the Lorenz plot section method applied to all variables showed a relatively high correlation, especially y as 0.904. Therefore, we can say that the Lorenz plot section method can detect the stretch-and-fold mechanism as clearly as the Poincaré section method.

We also examined if the Lorenz plot section method can detect the stretch-and-fold mechanism in high-dimensional attractor. We used a hyperchaotic Rössler system[7]. The equations are described by:

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + 0.25y + w, \\ \dot{z} = 3 + xz, \\ \dot{w} = -0.5z + 0.05w. \end{cases} \quad (4)$$

The results of applying the Lorenz plot section method applied to the variable y of the hyperchaotic Rössler system are shown in Fig.6. From Fig.6, we can see the entropy gradually getting higher, and then, lower. This characteristic is the same as the entropy change of the Poincaré section of the three-dimensional Rössler equation, and this result indicates that the stretch-and-fold mechanism of the high-dimensional system can also be detected.

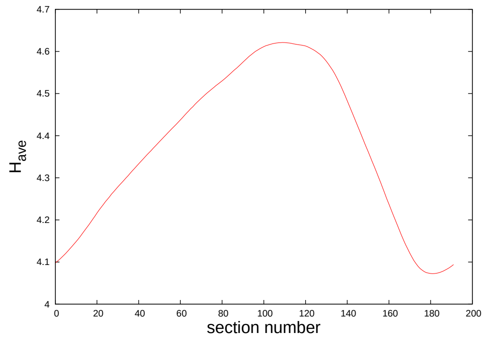


Figure 6: Entropy change of the Lorenz plot section applied to the variable y of the Rössler hyperchaotic system(Eq.(4)).

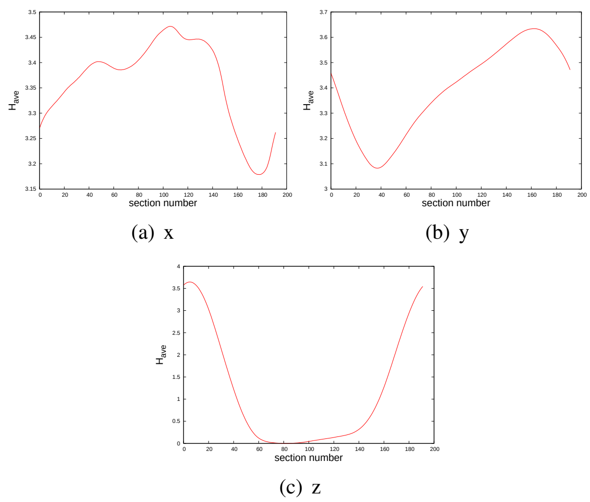
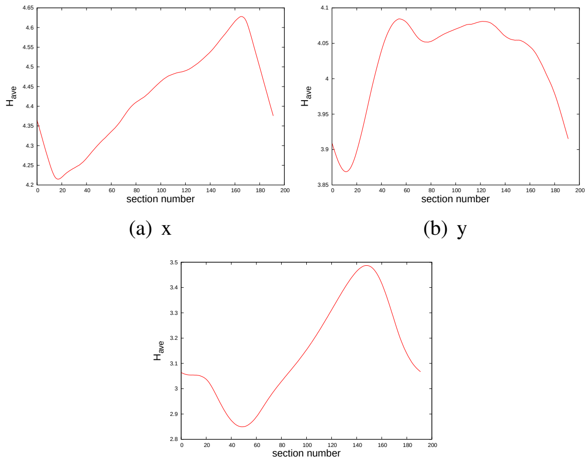


Figure 7: Entropy change of the Lorenz plot section method applied to all variables of the Rössler equations embedded in a three-dimensional state space. The average values $H_{\rm ave}$ of the entropy H of all sections are plotted with $1,000$ trials. The number of sections is 192 and $l = 50$.



(a) x (b) y

(c) z

Figure 8: Entropy change of the Lorenz plot section method applied to all variables of the Lorenz equations(Eq.(5)) embedded in a three-dimensional state space. The average values H_{ave} of the entropy H of all sections are plotted with $1,000$ trials. The number of sections is 192 and $l = 50$.

Next we will show the results of the Lorenz plot section method applied to an observed single variable time-series. Figures 7 and 8 show how the entropy H_{ave} change in case that the Lorenz plot section method is applied to all variables of the Rössler equations and the Lorenz equations[3]:

$$\begin{cases} \dot{x} = -10(x-y), \\ \dot{y} = -xz + 28x - y, \\ \dot{z} = xy - \frac{8}{3}z. \end{cases} \tag{5}$$

We reconstructed attractors in a three-dimensional state space for both cases. Qualitatively compared to the entropy change of the Poincaré section and the Lorenz plot section of the Rössler attractor, we can see the same characteristic of the entropy change.

5. Conclusions

We evaluated the performance of the Lorenz plot section method by calculating entropy of the points plotted on the sections. The results show that the Lorenz plot section method can assign sections semi-automatically on attractors and detect the stretch-and-fold mechanism clearly enough as compared to the Poincaré section method. Moreover, by using the entropy, the Lorenz plot section method can detect the stretch-and-fold mechanism of a high-dimensional chaotic dynamical system such as the hyperchaotic Rössler system. We also applied the method to reconstructed attractors from a chaotic time-series.

Although the proposed method show high quality, we have to show that our method can distinguish the entropy change of the chaotic dynamics from other dynamics, such as torus and noisy-periodic dynamics. It is also one of the important future works to apply the proposed method to real time series to identify the stretch-and-fold mechanism in the real world.

Acknowledgments

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References

[1] G. Robert et al, *Rev. Mod. Phys.*, vol.**70**, pp.1455–1529, 1998.

[2] H. Morioka et al, *Proc. IEICE Gen. Conf. '06*, **A-2-24**, p.60, 2006.

[3] N. Lorenz et al, *J. Atmos. Sci.*, Vol.**20**, pp.130–141,1963.

[a] M. Suefuji et al, *Proc. IEICE Soc. Conf. '11*, **A-2-8**, p.45, 2011.

[b] M. Suefuji et al, *Proc. IEICE gen. Conf. '12*, **A-2-25**, p.70, 2012.

[4] O. E. Rössler et al, *Phy. Lett.*, Vol.**57A**, pp.397–398, 1976.

[5] F. Takens et al, *Lecture Notes in Mathematics*, Vol.**898**, pp.366–381, 1981.

[6] A. M. Fraser et al, *Phy. Rev. A*, Vol.**33**, pp.1134–1140, 1986.

[7] O. E. Rössler et al, *Phy. Lett. A*, Vol.**71**, pp.155–157, 1979.