IEICE Proceeding Series

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Vol. 2 pp. 2-5
Publication Date: 2014/03/18
Online ISSN: 2188-5079

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Abstract—Since A. M. Turing introduced a notion of computability, various types of real number computation theory has been studied for past 80 years [1-6]. Some of them are of interest in nonlinear and statistical physics, and others are as extensions of mathematical theory of computation. In this review paper, we introduce recently developed computability theory of Julia sets in the framework of M. Braverman and M. Yampolsky [7], and give remarks and future works.

1. Introduction

Chaos and fractals have been studied from a viewpoint of computability in physical systems [4][3][6]. They are commonly based on their nature of complexity arising from iteration of simple rules. In this review paper, we introduce recently developed computability theory of Julia sets in the framework of M. Braverman and M. Yampolsky [7], and give remarks and future works in terms of computational complexity.

1.1. Classical definitions of computation

In this subsection, we show classical definitions of computable real numbers introduced by A. M. Turing [1] and computable real functions introduced by M. Pour-El [2].

Turing computability is a fundamental concept of computation [1]. It is defined by rather a physical automaton model, called Turing machine.

Def.1.1.1(Computability)
A function $f(x)$ is computable if there is a Turing machine M such that M takes x as an input and outputs $f(x)$.

We say that the number of steps $M(w)$ the Turing machine M makes before terminating with an input w is the running time. This is a basis of computational complexity theory.

Def.1.1.2(Time complexity)
For a Turing machine M on input w, the time complexity of M is the function $T_M : \mathbb{N} \to \mathbb{N}$ such that

$$T_M(n) = \max_{|w|=n}\{\textit{the running time of } M(w)\}.$$

A definition of computable real numbers is given by Turing.

Def.1.1.3(Computable real number)
A real number α is said to be computable if there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that

$$|\alpha - f(n)| < 2^{-n} \ (\forall n \in \mathbb{N}).$$

It is known that most real numbers are uncomputable. Examples of computable real number are e, π, and $\sqrt{2}$.

Later, a definition of computable real function is introduced by M. Pour-El in a context of computable analysis and constructive mathematics [2].

Def.1.1.4(Computable real function)
Let $I^q = \{a_i \le x_i \le b_i, 1 \le i \le q\} \subseteq \mathbb{R}^q$, where a_i and b_i are computable reals, be a computable rectangle. A real function $f : I^q \to \mathbb{R}^q$ is said to be computable if
(i) f is sequentially computable, i.e. f maps every computable sequence of points $x_k \in I^q$ into a computable sequence $\{f(x_k)\}$ of real numbers.
(ii) f is effectively uniformly continuous, i.e. there is a computable function $d : \mathbb{N} \to \mathbb{N}$ such that for all $x, y \in I^q$ and all N:

$$|x - y| \le \frac{1}{d(N)} \textit{ implies } |f(x) - f(y)| \le 2^{-N},$$

where $|\cdot|$ denotes Euclidean norm.

It is known that solutions of PDE described with computable real function may be uncomputable. Examples of such PDE is given by a class of wave equation.

1.2. Julia sets

In this subsection, we define Julia set in complex dynamical systems [8]. We will denote n-th iterate of mapping R by R^n. To say simply, the Riemann sphere is the union of complex plane and a point at infinity, i.e., $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let z_0 is a periodic point of period $n \in \mathbb{N}$. Then the periodic orbit of z_0 is called attracting or repelling if the derivative $|D^n R(z_0)| < 1$ or > 1.

Def.1.2.1(Fatou set and Julia set)
Let R be a rational function of degree $d \geq 2$ on the Riemann sphere $\hat{\mathbb{C}}$. The Fatou set is the set of points which have an open neighborhood $U(z)$ on which the family of iterates $R^n|_{U(z)}$ is equicontinuous. The Fatou set is denoted by $F(R)$. The open set $J(R) = \hat{\mathbb{C}} \setminus F(R)$ is called Julia set.

Def.1.2.2(Filled Julia set)
Let R be a rational function of degree $d \geq 2$. Then the filled Julia set $K(R)$ is defined as follows:

$$K(R) = \{z \in \hat{\mathbb{C}} | \sup_n |R^n(z)| < \infty\}.$$

Intuitively, a filled Julia set is a set of points which remain bounded under the iteration of R, and a Julia set is boundary of a filled Julia set.

Let $P_c(z) = z^2 + c$ and $J_c = J(P_c)$ as a class of Julia sets of polynomial with $d = 2$. For instance,when $c = 0$, the origin and the point at infinity are attracting points. Let z is in the interior of the unit disk $\mathbb{U}$. Then $P_0^n(z) \to 0$ as $n \to \infty$. So the family of iterates is equicontinuous. By the same argument, in the case that $z \in \hat{\mathbb{C}} \setminus \mathbb{U}$ the family of iterates is equicontinuous. But if z is on the unit circle, on the neighborhood of z the family of iterates cannot be equicontinuous. Thus the unit circle $\{z \in \hat{\mathbb{C}} ||z| = 1\}$ is the Julia set of $P_0(z)$. See examples of Figure 1.

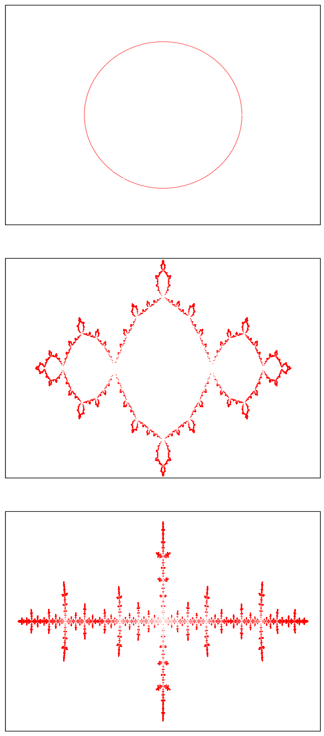

Figure 1: Julia sets J_c of $P_c(z) = z^2 + c$ with $c = 0$ (top), $c = -1$ (middle), and $c = -1.543689$ (bottom).

Def.1.2.3(Mandelbrot set)
Let $P_c(z) = z^2 + c$. The Mandelbrot set $\mathcal{M}$ is the set of parameters c which the orbit of the origin remains bounded.

Note that if $c \in \mathcal{M}$ then the Julia set is connected (is not Cantor set).

2. Computability of Julia sets

In this section, we consider the computability and complexity of Julia sets in the framework of M. Braverman and M. Yampolsky [7]. Let K be a compact subset of $\mathbb{R}^k$. In this framework, computable function must be continuous, so the characteristic function of $S \subset \mathbb{R}^k$, χ_S, is not computable unless $S = \emptyset$ or $\mathbb{R}^k$ itself.

To study computability of Julia sets, a geometric interpretation of Ko's computability based on Hausdorff metric is given by Braverman and Yampolsky.

2.1. Ko's definitions of computable real function

We introduce a framework of computable real numbers and functions by K. Ko [5]. A useful notion for computability of real number is a set of dyadic numbers $\mathbb{D}$ which is defined by

$$\mathbb{D} = \left\{ \frac{k}{2^l}; k \in \mathbb{Z}, l \in \mathbb{N} \right\}.$$

Now we define an oracle approximating a real number with precision n in finite time steps.

Def.1.3.1(Oracle)
A function $\phi : \mathbb{N} \to \mathbb{D}$ is called an oracle for $x \in \mathbb{R}$ if it satisfies $|\phi(m) - x| < 2^{-m} (\forall m)$.

Intuitively, oracle is a "equipped device" for computers and they cannot be described as an algorithm. We will denote a Turing Machine with an oracle for a real number x by M^ϕ. When we write $M^\phi(n)$, n represents precision of approximation of x.

Def.1.3.2(Computability of function)
Let $S \subset \mathbb{R}$ and let $f : S \to \mathbb{R}$. Then f is said to be computable if there is an oracle Turing machine $M^\phi(n)$ such that the following holds. If $\phi(m)$ is an oracle for $x \in S$, then for all $n \in \mathbb{N}$, $M^\phi(n)$ returns $q \in \mathbb{D}$ such that $|q - f(x)| < 2^{-n}$.

Note that discontinuous functions cannot be computable such as Heaviside functions with this definition.

Def.1.3.3(Polynomial time computability of real function)
Let $S \subset \mathbb{R}^k$ and p be some polynomial. A function $f : S \to \mathbb{R}$ is said to be polynomial time computable if there is a machine M^ϕ computing it such that $T_M(n) \le p(n)$.

Now we introduce Braverman-Yampolsky computability. Intuitively, this is a computability concept based on "drawing" a picture of K with round pixels on the computer screen.

Def.2.1.1(Regular computability)
The set $K \subset \mathbb{R}^k$ is computable if a Turing machine M computing a function $f_K(d, r)$ from the family

$$f_K(d,r) = \begin{cases} 1 & \text{if } B(d,r) \cap K \neq \emptyset \\ 0 & \text{if } B(d,2r) \cap K = \emptyset \\ 0 \text{ or } 1 & \text{otherwise} \end{cases}$$

exists.

A schematic view is given in Figure 2.

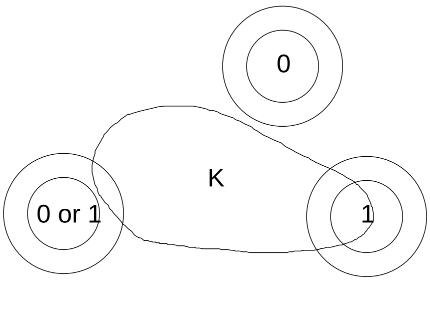

Figure 2: Schematic view of regular computability.

Def.2.1.2(Weak computability)
The set $K \subset \mathbb{R}^k$ is weakly computable if there is an oracle Turing Machine $M^\phi(n)$ such that, if $\phi = (\phi_1, \phi_2, \ldots, \phi_k)$ represents $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, then the outputs of $M^\phi(n)$ is

$$M^\phi(n) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } B(x, 2^{-(n-1)}) \cap K = \emptyset \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases}$$

A schematic view is in Figure 3. The value of $M^\phi(n)$ is not 1 unless the center of pixel is contained in K.

We rewrite the definition of regular computability to those for Julia sets.

Def.2.1.3(Regular computability of set-valued function)

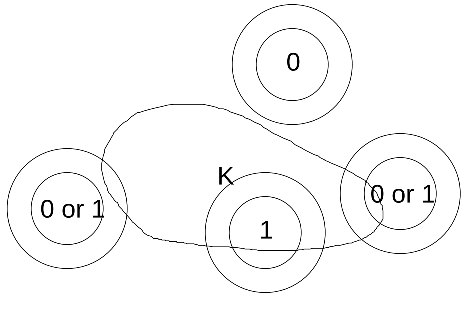

Figure 3: Schematic view of weak computability.

Let $S \subset \mathbb{R}^k$, and $F : S \to K_2^*$ be a function which maps a points in S to K_2^* which is compact subsets of $\mathbb{R}^2$. Then F is said to be computable on S if there is an oracle Turing machine $M^{\phi_1,\ldots,\phi_k}(d,r)$ which, for the oracles representing a point $x = (x_1, \ldots, x_k) \in S$, computes a function $f^{\phi_1,\ldots,\phi_k} : \mathbb{D}^2 \times \mathbb{D} \to \{0,1\}$ from the family

$$f^{\phi_1,\ldots,\phi_k}(d,r) = \begin{cases} 1 & \text{if } B(d,r) \cap F(x) \neq \emptyset \\ 0 & \text{if } B(d,2r) \cap F(x) = \emptyset \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases}$$

The definitions of "regular" and "weak" computability are different from each other. However, they produce same results in terms of computability. As for computational complexity, we can define time complexity as following: The running time $T_M(n)$ is the longest time it takes to compute $f^{\phi_1,\ldots,\phi_k}(d,r)$ where $r = 2^{-n}$ and $d \in (\mathbb{Z}/2^{2n})$. In terms of computational complexity, the definitions of regular and weak computability may produce different results. When we adopt the regular computability to consider our Julia set problems, then the following theorems hold. [7].

A rational mapping $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called *hyperbolic* if the orbit of every critical point of R is either periodic, or converges to an (super-)attracting cycle.

Thm.2.1.4(Computability of hyperbolic Julia sets)
Fix $d \ge 2$. There exists a Turing machine M^ϕ with oracle access to the coefficients of a rational mapping of degree d which computes the Julia sets of every hyperbolic rational map of degree d. Moreover, the Julia sets can be computed in polynomial time.

Thm.2.1.5(Uncomputable Julia sets)
There exists computable values of the parameter c, such that the Julia set J_c is not computable by a Turing machine M^ϕ with oracle access to c.

A practical consequence of the existence of uncomputable Julia sets J_c is that we will never see their pictures on computer screen. On the other hand, in the case of $d = 2$, Theorem2.1.4 and the hyperbolicity conjecture, which states that hyperbolic parameters are dense in the Mandelbrot set $\mathcal{M}$, asserts most Julia sets are computable.

The computability of filled Julia sets are simpler than that of Julia set. We denote filed Julia set of polynomial p by K_p.

Thm.2.1.6(Computability of filled Julia set)
For any polynomial $p(z)$ there is an oracle Turing machine $M^{\phi}(n)$ that, given an oracle access to the coefficients of $p(z)$, outputs 2^{-n}-approximation of the filled Julia set K_p.

3. Future works

This research will be developed to analyze complexity of computable Julia sets. Extending known results for real dynamical systems in nonlinear and statistical physics may shed lights to complexity of nonlinear phenomena. Applications to cryptographic systems and formal language theory with this frameworks are promising future works.

Acknowledgements

The authors thank to Dr. Tomoo Yokoyama (Hokkaido University) for useful discussion.

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