ANALYSIS OF WAVE FORM OF BACKSCATTERED MICROWAVE PULSE BASED ON A PROBABILISTIC MODEL OF OCEAN WAVES

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1 Introduction

A satellite- or air-borne radar altimeter plays an important part for the survey of ocean surface and ocean dynamics. An altimeter emits a train of microwave pulses onto the ocean surface, and from the received pulse echoes various parameters are estimated, such as the average height of waves, wave directions, etc.. This paper gives a theoretical anlysis for the wave form of the received pulse echoes based on a stochastic model of random ocean surface.If the microwave wavelength is short compared to the wave height and sea wave is reasonably smooth, we may use the physical optics approximation for the scattering [1]: That is, the microwave illuminated from a satellite is mostly reflected from extremal points of the sea surface,i.e., crests, troughs and saddles.; and the reflection coefficient is inversely proportional to the Gaussian curvature at each point.

Our analysis goes as follows: We assume that the sea surface is described by a homogeneous Gaussian random field, and firstly obtain the joint probability distribution for the three kinds of extrema and the principal curvatures of the random surface; this is obtained by extending S.O.Rice's method for the maxima of a stationary process [2] to the case of a homogeneous random surface. Second, we represent the scattered radio waves as a sum of the reflected waves from those stationary points by means of the stationary phase method. As the beam radius at the sea level is usually large enough to contain many stationary points, so that, invoking the law of large numbers the sum of reflections from those points can be replaced by its probabilistic average using the above probability density. In this manner we can theoretically calculate the echoed pulse form from a stochastic sea surface. Finally we show some numerical results calculated using a suitable spectrum for the ocean waves.

2 Probability Distribution for Extrema of Random Surface

We assume a random surface $z = f(\boldsymbol{r})$, $\boldsymbol{r} = (x, y)$, is a homogeneous Gaussian random field, with 0 mean and the spectral density $S(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} = (\lambda, \mu)$. Let the partial derivatives of f; $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$, be denoted by u, v, r, s, t, respectively, and put $\boldsymbol{z} \equiv (z, r, s, t)$ and $\boldsymbol{u} \equiv (u, v)$. Then the 6-dimensional Gaussian density for $(\boldsymbol{z}, \boldsymbol{u})$ can be written as a product of two independent Gaussian densities $p_1(\boldsymbol{z})$ and $p_2(\boldsymbol{u})$ associated with $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, where $\boldsymbol{R}_1$ is the 4×4 covariance matrix of $\boldsymbol{z}$, and $\boldsymbol{R}_2$ the 2×2 covariance matrix of $\boldsymbol{u}$, respectively:

$$p(z, r, t, s, u, v) \;=\; p_1(z, r, t, s) p_2(u, v) \tag{1}$$

By the coordinate rotation $(x, y) \rightarrow (x', y')$ by an angle θ, we have the transformations $(u, v) \rightarrow (u', v')$, and $(r, t, s) \rightarrow (r', t', s')$. Particularly if $\tan 2\theta = 2s/(r - t)$, we have $(r', t', s') = (r_0, t_0, 0)$, and at an extremal point where $u = v = 0$, r_0 and t_0 give the two principal curvatures and satisfy the equalities: $r_0 + t_0 = r + t$ and $r_0 t_0 = rt - s^2$, the latter

being the Gaussian total curvature. Maxima, minima and saddles occur in the regions specified by the following inequalities, respectively: $R_{\min}$, $r_0 > 0$, $t_0 > 0$; $rt - s^2 > 0$, $r > 0$; $R_{\max}$, $r_0 < 0$, $t_0 < 0$; $rt - s^2 > 0$, $r < 0$; R_{sad}, $r_0 t_0 < 0$; $rt - s^2 < 0$. By the transformation $(r, t, s, u, v) \to (r_0, t_0, \theta, u', v')$ we can rewrite

$$\begin{aligned} & p_1(z, r, t, s) p_2(u, v) \mathrm{d}r \mathrm{d}t \mathrm{d}s \mathrm{d}u \mathrm{d}v \\ & = p_1(z, r_0 \cos^2\theta + t_0 \sin^2\theta, r_0 \sin^2\theta + t_0 \cos^2\theta, (r_0 - t_0)\sin\theta\cos\theta) \\ & \quad \times p_2(u'\cos\theta - v'\sin\theta, u'\sin\theta + v'\cos\theta)\,|r_0 - t_0|(1 + \sin^2 2\theta)\mathrm{d}r_0 \mathrm{d}t_0 \mathrm{d}\theta \mathrm{d}u' \mathrm{d}v' \\ & \equiv p_3(z, r_0, t_0, \theta; u', v') \mathrm{d}r_0 \mathrm{d}t_0 \mathrm{d}\theta \mathrm{d}u' \mathrm{d}v' \end{aligned} \tag{2}$$

To this form of probability distribution we can apply Rice's method [2]: the probability that the random field has an extremum within a small volume $\mathrm{d}z\mathrm{d}x\mathrm{d}y$ at the point (x,y) with the parameters (r_0, t_0, θ) in an infinitesimal region $\mathrm{d}r_0\mathrm{d}t_0\mathrm{d}\theta$ is given by

$$\mathrm{d}z \int_{|r_0|\mathrm{d}x'} \mathrm{d}u' \int_{|t_0|y'} \mathrm{d}v' p_3(z, r_0, t_0, \theta; u', v') = \mathrm{d}z\mathrm{d}x\mathrm{d}y |r_0 t_0| p_3(z, r_0, t_0, \theta; 0, 0) \tag{3}$$

times $\mathrm{d}r_0\mathrm{d}t_0\mathrm{d}\theta$, where $\mathrm{d}x'\mathrm{d}y' = \mathrm{d}x\mathrm{d}y \equiv \mathrm{d}\boldsymbol{r}$, and $r_0 t_0 = rt - s^2$ is the Gaussian curvature. From (3) we can derive various probability distributions by integrating it over an appropriate region mentioned above.

3 Backscattered Microwave from Ocean Surface

First we assume typical dimensions for the satellite radar altimeter. The antenna (satellite) is located at $(z_0, x = y = 0)$, $z_0 \simeq 800$ km, and denote the sea surface by $z = f(x, y)$ where the wave height is roughly $|\, z \,| \leq 10$m. Let the microwave wavelength be $\lambda = 2$cm, the beam pattern be described by $b(\rho^2)$, $\rho \equiv \sqrt{x^2 + y^2}$, and the beam radius be roughly 3km. The beam angle is then roughly $3/800 = 0.2$deg. The distance from antenna $(0, 0, z_0)$ to a scattering point (x, y, f) on the sea surface is written $r \simeq z_0 - f + \rho^2/2z_0$, and the backscattered field of microwave with wave-number k , received at the satellite antenna, can be approximately written

$$E(k) \;\simeq\; \frac{A}{z_0^2} \mathrm{e}^{-2\mathrm{i}kz_0} \int\int_S \mathrm{e}^{2\mathrm{i}kg(x,y)} b(\rho^2) \mathrm{d}x\mathrm{d}y \tag{4}$$

where S denotes the beam area, A the amplitude and $g \equiv f - \rho^2/2z_0$. Let the stationary point of g be $\boldsymbol{r}_n, n = 1, 2, .,$ and partial derivatives of g at $\boldsymbol{r}_n$ can be approximately put, e.g., $g_{xx}(\boldsymbol{r}_n) \simeq f_{xx}(\boldsymbol{r}_n) = r_n$, if we neglect the curvature of spherical phase front compared to those of ocean waves. Then the Gaussian curvature at $\boldsymbol{r}_n$ can be written as $H_n = r_n t_n - s_n^2$. By the stationary phase method, the integral of (4) is mostly contributed from the points of its stationary phase [1], and we have

$$E(k) \sim \frac{\pi}{k^2} \frac{A}{z_0^2} \mathrm{e}^{-2\mathrm{i}kz_0} \sum_{n=1}^{M} \frac{\sigma_n k}{|H_n|^{1/2}} b(\boldsymbol{r}_n) \mathrm{e}^{-\mathrm{i}k\frac{\rho_n^2}{z_0}} \mathrm{e}^{2\mathrm{i}kf(\boldsymbol{r}_n)} \tag{5}$$

where σ_n equals $i, -i, 1$ depending on $\boldsymbol{r}_n$ being max., min. or saddle point, respectively, and M denotes the number of extremal points in the beam-illuminated area. This implies that the staionary point $\boldsymbol{r}_n$ corresponds to the point of reflection, and $\sigma_n / \mid H_n \mid^{1/2}$ gives the reflection coefficient.

Let the transmitted pulse be given by $\mathrm{e}^{i\omega_0 t} a(ct)$, where $k_0 c = \omega_0$ gives the angular frequency of the microwave, and $a(ct)$ denotes the real pulse form of the amplitude. Then

the pulse waveform of the scattered wave can be given

$$E(t) = \frac{\pi}{k_0^2 z_0^2} e^{j\omega_0 t} \sum_{n=1}^{M} \frac{k_0 \sigma_n b(\boldsymbol{r}_n)}{|H_n|^{1/2}} a\left(ct - 2z_0 - \frac{\rho_n^2}{z_0} + 2f(\boldsymbol{r}_n)\right) \tag{6}$$

If f is much larger than the wavelength λ, the phase shift φ_n of $\sigma_n \equiv e^{i\varphi_n}$, equivalent to time delay or phase length $\pm\lambda/4$ or 0, can be neglected in the statistical sum of (6).

Suppose the scattered wave (6) be coherently detected to get the amplitude of reflected pulse. The correlation length of f is much shorter than the beam radius D, so that M is very large. By the law of large numbers, the sum of variables with identical probability distributions can be replaced by its probabilistic average multiplied by average number of points. Therefore, if the detected pulse amplitude is suitably normalized, we may replace the statistical sum of (6) by its average. Thus, discarding the constant factor, we can write

$$|E(t)| \simeq \int_S b(\boldsymbol{r}) \bar{a}\left(ct - 2z_0 - \frac{\rho^2}{z_0}\right) \mathrm{d}\boldsymbol{r} \tag{7}$$

$$\bar{a}(ct) \equiv \left\langle \frac{k_0}{|H|^{1/2}} a(ct + 2f(\boldsymbol{r})) \right\rangle \equiv k_0^2 \int_{-\infty}^{\infty} a(ct + 2z) q(z) \mathrm{d}z \tag{8}$$

$$q(z) \equiv \frac{1}{k_0} \int\int\int_{-\infty}^{\infty} |rt - s^2|^{1/2} p(z, r, t, s; 0, 0) \mathrm{d}r \mathrm{d}t \mathrm{d}s \tag{9}$$

where $\mathrm{d}\boldsymbol{r} = \mathrm{d}x\mathrm{d}y = \rho \mathrm{d}\rho \mathrm{d}\theta$, and we have regarded $a(ct)$ and $b(\boldsymbol{r})$ as real-valued functions. Particularly, if the original pulse form is an ideal one; $a(ct) = A\delta(ct)$, (pulse width 0), then the averaged scattered pulse (8) has the same form with $q(z)$: $\bar{a}(ct) = q(-ct/2)k_0^2/2$. The integration in (7) implies that the averaged pulse $\bar{a}(ct)$ is further broadened by the spherical time delay due to the beam spread. If the beam is uniform; $b = 1, 0 < \rho < D$, we have

$$|E(t)| = 2\pi \int_0^D \bar{a}\left(ct - 2z_0 - \frac{\rho^2}{z_0}\right) \rho \mathrm{d}\rho \tag{10}$$

4 Numerical Examples for Scattered Pulse Wave Form

To calculate the probability distribution for the extrema and the curvatures, we need to a model spectrum for the ocean waves. For example, we may take an isotropic spectrum for an omni-directional wave model:

$$S(\Lambda) = R/(2\pi K_m)\Lambda^{2m} e^{-\ell^2\Lambda^2}, \quad m = 0, 1, 2, \cdots \tag{11}$$

where $\Lambda \equiv \sqrt{\lambda^2 + \mu^2}$, R denotes the variance of the wave height and $K_m \equiv m!/2\ell^{2(m+1)}$ the normalizing factor. The spectral density (11) has a peak at $\Lambda_p \equiv \sqrt{m}/\ell$, which describes the average wave length $2\pi/\Lambda_p$ of ocean waves (Fig.1, spectral peak position normalized as $\Lambda_p = 1$). Figs.2 and 3 show an example for $P_{\min}$ and P_{saddle}, and Fig.4 a joint probability density $p_{\min}(z, 1/t_0 r_0)$. Fig.5 shows $q(z)$ for the average scattered pulse form and Fig.6 the echo pulse for the Gaussian beam with beam radius 3 km.

Reference

[1] H.C.Chen: Theory of Electromagnetic Waves, McGraw-Hill, (1983)

[2] S.O.Rice: Matheomatical Analysis of Random Noise, B.S.T.J. Vol.23, pp.282-332 (1944), Vol.24, pp.46-156 (1945)

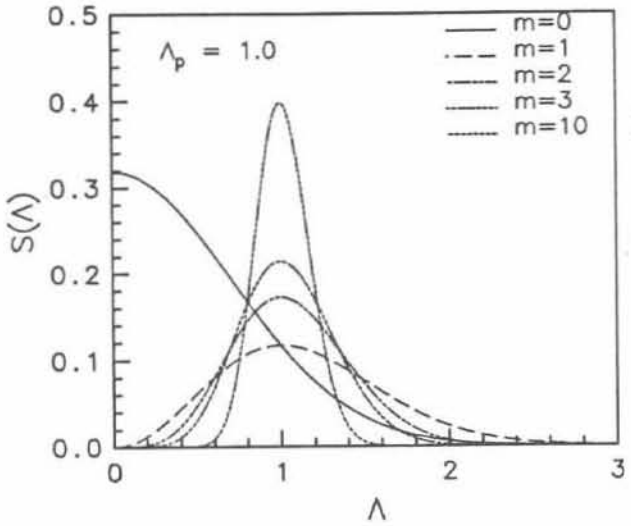

Fig.1 Spectral Density

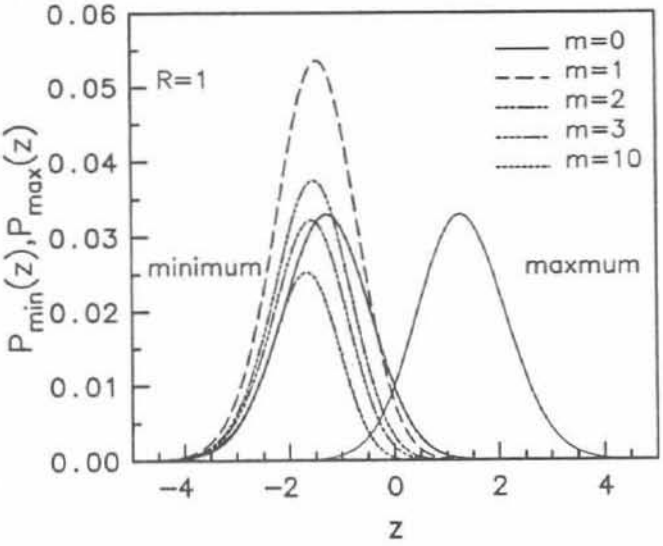

Fig.2 $P_{\min}(z)$, $P_{\max}(z)$

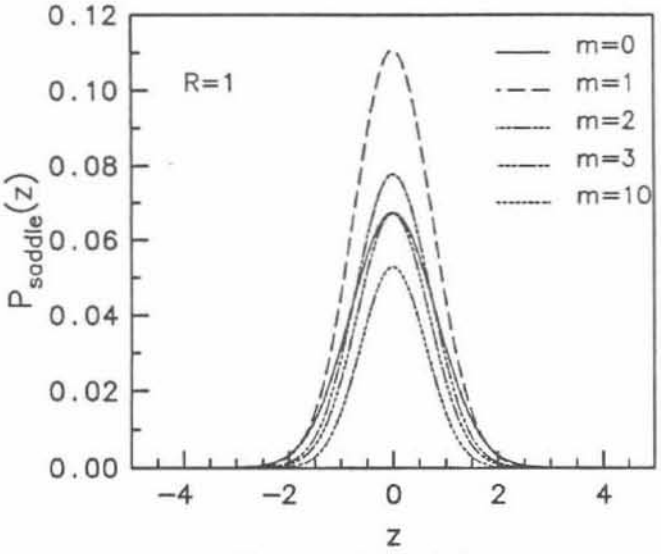

Fig.3 $P_{\text{saddle}}(z)$

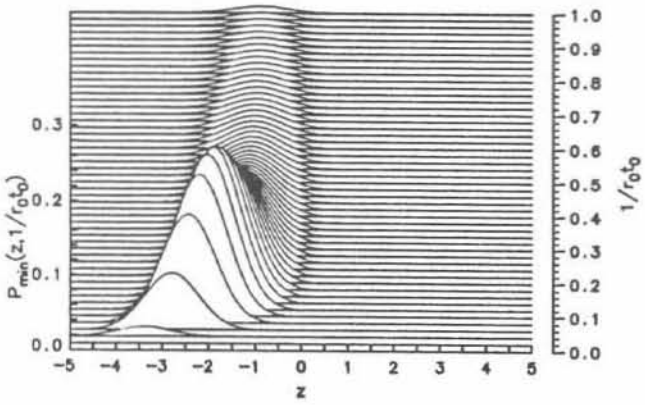

Fig.4 $P_{\min}(z, 1/t_0r_0)$

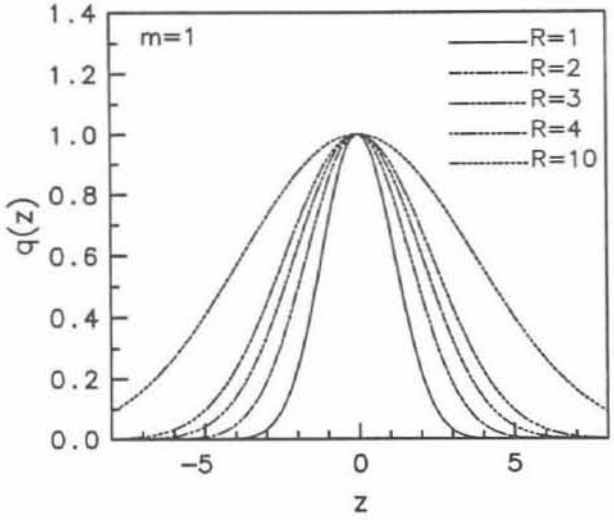

Fig.5 $q(z)$

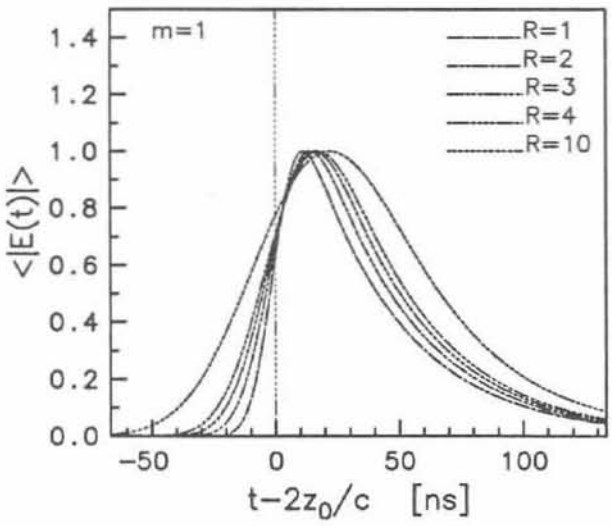

Fig.6 Pulse Echo