

# A Uniform Asymptotic Solution for Lateral Displacement of a Gaussian Beam at a Dielectric Interface

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## 1. Introduction

The problems of a Gaussian beam that is incident on a plane dielectric interface from a denser medium to a rarer one have been the important research subjects studied by many researchers [1]-[5]. By applying the Fresnel approximation to the integral representation for the reflection of the beam at the interface, Horowitz and Tamir [3], [4] have derived the solution for the lateral displacement of the beam for incident angles that are close to the critical angle of the total reflection. Later on, Kozaki and Sakurai [5] have derived the approximate result for the arbitrary incident angles and compared with the exact values calculated numerically from the integral representation.

In this work, we shall derive a novel uniform asymptotic solution for the lateral displacement of a Gaussian beam at the plane dielectric interface. By comparing with the exact numerical solution, we will confirm the validity of the uniform solution proposed in the present study.

## 2. Formulation and Integral Representation for Scattering of Beam

Fig.1 shows the scattering of the Gaussian beam by the plane dielectric interface, the coordinate system  $(x, y, z)$ , and the beam coordinate systems  $(x_i, z_i)$  and  $(x_r, z_r)$ . The Gaussian beam is incident from the upper medium 1 with a dielectric constant  $\epsilon_1$  to the lower medium 2 with the dielectric constant  $\epsilon_0$ . The refractive index  $n$  defined by  $n = \sqrt{\epsilon_0 / \epsilon_1}$  is assumed to be  $n < 1$ .

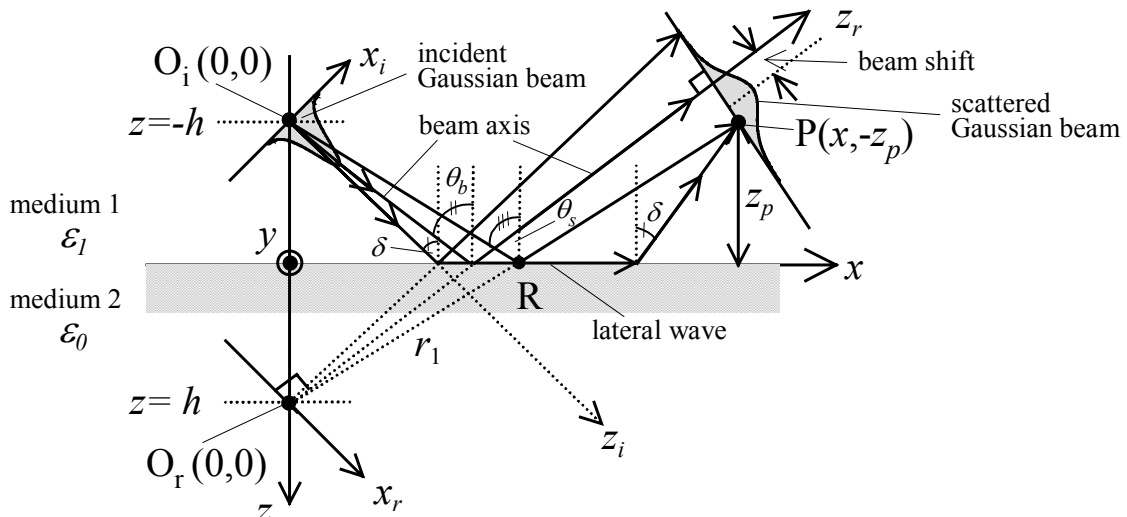


Fig 1. Schematic figure for scattering of beam by the plane dielectric interface, the coordinate system  $(x, y, z)$ , and the beam coordinate systems  $(x_i, z_i)$  and  $(x_r, z_r)$ .  $O_i(0, 0)$ : origin of  $(x_i, z_i)$ ,  $O_r(0, 0)$ : origin of  $(x_r, z_r)$ .

We assume the incident Gaussian beam polarized in  $y$  direction as follows



When the high-frequency condition  $k_1 r_1 \gg 1$  is satisfied, the integral  $E_y^{SDP}$  can be evaluated asymptotically by applying the saddle point technique which is applicable uniformly as  $\theta_s \rightarrow \delta$  [7], [8]. While the integral  $E_y^B$  around the branch cut  $C_B$  can also be evaluated asymptotically by deforming the integration contour  $C_B$  (see Fig.3) into the new steepest descent path and assuming  $\theta_s \rightarrow \delta$  [6]-[8]. Due to the limitation of the space, here we will give the final result:

$$E_y^r = E_y^{go} + U(\theta_s - \delta) E_y(C_B) + E_y^{tran} \quad (5)$$

where  $E_y^{go}$  denotes the geometrically reflected beam and  $E_y(C_B)$  denotes the laterally shifted beam. The last term  $E_y^{tran}$ , defined as the ‘‘transition term’’, plays an important role only in the transition region near the critical angle (i.e.,  $\theta_s = \delta$ ). Far away from the transitional region, the last term vanishes as the value  $|\theta_s - \delta|$  increases.  $E_y^{go}$ ,  $E_y(C_B)$ , and  $E_y^{tran}$  are defined as follows:

$$E_y^{go} = A(k_1 r_1) \Gamma(\theta_s) B(\theta_s, \theta_b) \quad (6)$$

$$A(k_1 r_1) = \{k_1 / 2\pi r_1\}^{1/2} \exp(ik_1 r_1 - i\pi/4) \quad (6a)$$

$$B(\theta_s, \theta_b) = \exp\left[-\left\{\frac{k_1 W}{2} \sin(\theta_s - \theta_b)\right\}^2\right] \cos(\theta_s - \theta_b) \quad (6b)$$

$$E_y(C_B) = A(k_1 r_1) \left[ \frac{\sin^2 \delta}{\sin^2 \theta_b} \frac{4\pi r_1}{\exp(ik_1 r_1)} \frac{n}{2\pi m} \left(\frac{k_1}{r \cos \delta}\right)^{1/2} \right. \\ \left. \cdot \exp\left(ik_1 r_1 \cos \alpha + i\eta^2/2\right) \frac{D_{-3/2}(\eta + i\eta)}{(k_1 r_1 \cos \alpha)^{3/4}} \right] \cdot B(\delta, \theta_b), \quad \alpha = \theta_s - \delta \quad (7)$$

$$E_y^{tran} = A(k_1 r_1) \Gamma^{(2)}(\theta_s) \sqrt{n - \sin \theta_s} \left\{ \frac{D_{1/2}(\eta - i\eta)}{2^{1/4} \eta^{1/2} e^{-i\pi/8} e^{i\eta^2/2}} - 1 \right\} B(\theta_s, \theta_b) \quad (8)$$

$$\Gamma^{(2)}(\theta_s) = \frac{-2m \cos \theta_b \sqrt{n + \sin \theta_b}}{m^2 \cos^2 \theta_b + (\sin^2 \theta_b - n^2)} \quad (8a)$$

$$\eta = \sqrt{2k_1 r_1} \sin \frac{\delta - \theta_s}{2} \quad \text{for } \theta_s < \delta, \quad \eta = \sqrt{2k_1 r_1} \sin \frac{\theta_s - \delta}{2} \quad \text{for } \theta_s > \delta \quad (8b)$$

When the observation point is located far away from the transition region,  $\eta$  defined by (8b) takes the large values (i.e.,  $\eta \gg 1$ ). Then the parabolic cylinder function  $D_{1/2}(\eta - i\eta)$  [7]-[9] is approximated by

$$D_{1/2}(\eta - i\eta) \sim 2^{1/4} \eta^{1/2} e^{-i\pi/8} e^{i\eta^2/2} \quad (9)$$

Therefore the terms inside the parentheses  $\{ \}$  in (8) approach zero as the value of  $\eta$  increases. This is the reason why  $E_y^{tran}$  in (5) and (8) is defined as the ‘‘transition term’’.

## 4. Numerical Results and Discussions

In Fig.4(a), we have calculated the beams reflected on the dielectric interface for two beam angles  $\theta_b = 30^\circ$  and  $\theta_b = 70^\circ$  (see Fig.1). The numerical parameters used in the calculations are shown in the figure. The critical angle  $\delta$  is  $\delta = 60^\circ$  ( $\delta = \sin^{-1} n$ ,  $n = 0.866$ ). The solid curves (—) are calculated numerically from (2) and serve as the reference solutions. The open circles (ooo) for  $\theta_b = 30^\circ$  and  $\theta_b = 70^\circ$  in Fig.4(a) are calculated from  $E_y^{go}$  in (6). The solutions (ooo) calculated by using only the geometrical term  $E_y^{go}$  in (5) agree very well with the reference solutions (—). It is clarified that when the beam angle  $\theta_b$  (see Fig.1) is not close to the critical angle  $\delta = 60^\circ$ , the beam

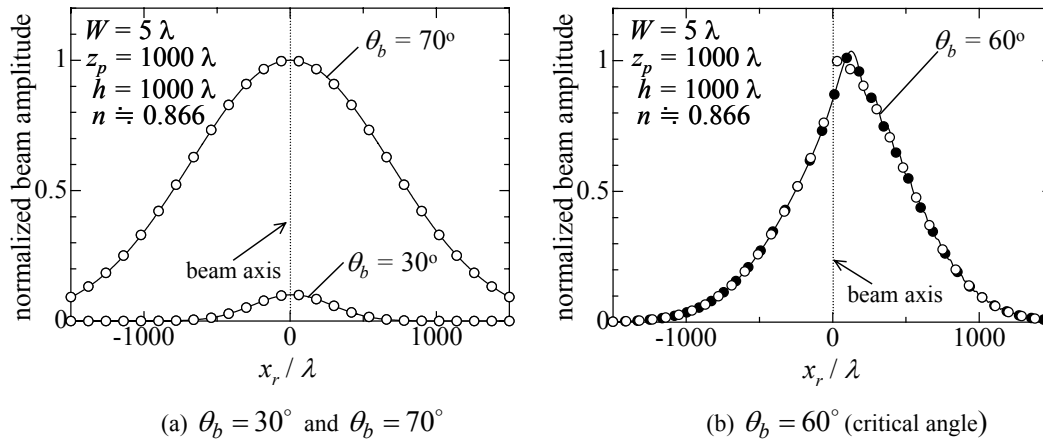


Fig.4 Numerical calculations of Gaussian beams scattered at the dielectric interface.

reflected on the dielectric interface can be obtained only from  $E_y^{go}$  derived in (6) ~ (6b).

The scattered beam for the beam angle  $\theta_b$  which is equal to the critical angle  $60^\circ$  is calculated in Fig.4(b). Now the beam calculated from  $E_y^{go}$  (ooo) in (6) shows the large deviation from the reference solution (—) near the beam axis where  $x_r/\lambda = 0$ . While, the beam calculated from (5) shown by the closed circles (●●●) agrees very well with the reference solution. It is observed that the beam is shifted to the  $+x_r$  direction. This phenomenon is well known as the Goos-Hänchen shift[1]-[5].

## 5. Conclusion

We have derived the uniform asymptotic solution for the Gaussian beam scattered at the plane dielectric interface. We have shown that the asymptotic solution, which uses the parabolic cylinder functions, agrees very well with the reference solution calculated numerically. Also shown is the Goos-Hänchen shift appeared for the beam angle  $\theta_b$  close to the critical angle  $\delta$ .

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