PROCEEDINGS OF ISAP '92, SAPPORO, JAPAN

SINGLING OUT PHYSICAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS IN ONEDIMENSIONAL PERIODIC GRATINGS OF WAVE DIFFRACTION THEORY

Yu.K.Sirenko
Institute of Radiophysics and Electronics of the Ukrainian Academy of Sciences, 12 Acad. Proskura st., Kharkov, 310085, the Ukraine

1. The classic methods of singling out the unique solution of the boundary value problem

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} + \kappa^2\varepsilon(y,z)\right] \mathcal{U}(y,z) = f(y,z),\ \kappa>0,\ g=\{y,z\}\notin \mathrm{int}\, S \quad (1)$$

$$\mathcal{U}(g)\Big|_S = 0 \quad (2)$$

corresponding to the stationary (time-dependence $\exp(-i\omega t)$) E-polarized (H-polarized waves may be considered in the same way) wave diffraction on onedimensional-periodic metal-dielectric gratings are investigated. Here $\kappa=\omega(\varepsilon_0\mu_0)^{1/2}$; the permittivity ε_0 , the permeability μ_0 -sealar parameters, characterizing the medium; $\mathcal{U}(g)$ - unique nonzero component of electric field strength E_x ; S - smooth contour of metallic grating element cross-section. Sufficiently smooth functions $1-\varepsilon(g)$: $\mathrm{Re}\,\varepsilon>0,\ \mathrm{Im}\,\varepsilon \geq 0$, $\varepsilon(y+2\pi,z)=\varepsilon(y,z)$ and $f(g)$, characterizing correspondingly the relative permittivity ε of the material, the grating is made of grating media and the source $f(g)$, are finite in arbitrary strip $R=\{g: 0\leq y\leq 2\pi,\ |z|<\infty\}\setminus \mathrm{int}\, S$. Their bearers are contained in $Q=\{g\in R: |z|\leq 2\pi\delta\}$.

2. Let us distinct two principally different methods of grating excitation: the quasiperiodic grating excitation with source function $f(y+2\pi,z)=\exp(i2\pi\Phi)f(y,z)$, $\mathrm{Im}\,\Phi=0$, and local source excitation with bearer $f(g)$ limited (finite) not only along z axis direction but along y axis too. In the first case, after the condition (2) was supplemented by relation

$$\mathcal{U}\left\{\frac{\partial \mathcal{U}}{\partial y}\right\}(2\pi,z) = e^{i2\pi\Phi}\,\mathcal{U}\left\{\frac{\partial \mathcal{U}}{\partial y}\right\}(0,z)$$

the problem (1),(2) may be considered in the strip (Floquet's channel). The radiation condition

$$\mathcal{U}(g)=\sum_{n=-\infty}^{\infty}\begin{Bmatrix} a_n \\ b_n\end{Bmatrix} e^{i[\Phi_n y \pm \Gamma_n(z\mp 2\pi\delta)]},\quad z \gtrless \pm 2\pi\delta \quad (3)$$

$$\Phi_n = n+\Phi\ ;\ \Gamma_n=(\kappa^2-\Phi_n^2)^{1/2},\ \mathrm{Im}\,\Gamma_n\geq 0,\ \mathrm{Re}\,\Gamma_n\geq 0.$$

conformed with physically justified requirement that there should not be waves coming from infinity is usually applied in the solution of such problems. Correct mathematical proof of corresponding principle was carried out only for the case of absorpting media /1/. When $\mathrm{Im}\,\varepsilon_0=0$ the following is valid

Theorem 1 (radiation principle). Let $\mathrm{Im}\,\varepsilon(g)>0$ for every g belonging to arbitrary nonzero measure set in Q. Then the condition (3) singles out the solution of (1),(2) for all $k > 0$. If $\mathrm{Jm}\,\varepsilon(g) \equiv 0$, then the solution of (1)-(3) exists and is unique for all $k > 0$, except at most the countable set $\{\overline{K}_n\}$ without finite accumulation points. The solution of the problem (1)-(2) can be received only if the following relation is valid (if $K = \overline{K} \in \{\overline{K}_n\}$)

$$\int_Q \mathrm{Res}\; G(g, g_0, \overline{K}, \Phi) \equiv 0. \tag{4}$$

The proof of this theorem based on the results /2,3/ enables us to consider Green function $G(g, g_0, K, \Phi)$ of boundary value problem (1)-(3) as meromorphous function of parameter k, varying on infinitely folded surface H, the first sheet of which (further let us call it simply k plane) is fully determined by values $\Gamma_n(K)$, $n=0,+1,...$ with real k and by the form of correspondent slits. The branching points $K_n : \Gamma_n(K_n) = 0$ are of the second order, real and coincide with those in which one of partial components of spatial scattering field spectrum (one of partial components in (3)) is slipping. If we denote by Ω the set of points $\overline{K} \in H$ that are the poles of $G(K)$, then on $H \setminus \Omega$ the unique solution of (1)-(3) exists, while $\mathrm{Res}\,G(\overline{K})$ defines nontrivial solution of (1)-(3) in the point $\overline{k}$ when $f(g) \equiv 0$ (here and further we assume that all the poles of $G(K)$ are simple). The condition $\mathrm{Jm}\,\varepsilon(g) > 0$ with assumed time-dependence means that energy absoption exists in nonideal dielectric of the gratings. The statement (4) with taking into consideration $G(g, g_0, K, \Phi) = G(g_0, g, K, -\Phi)$ /3/, may be reformulated in the terms of orthogonality of source function $f(g)$ and eigen functions of corresponding homogeneous problem.

3. On applying the limitary attenuation principle, the solution of (1),(2) (quasi-periodical excitation) is presented

$$\mathcal{U}(g, \mathrm{Re}\,K) = \lim_{\mathrm{Jm}K \to 0} \overline{\mathcal{U}}(g, K), \quad 0 < \arg K < \pi/2 \tag{5}$$

the limit is considered as convergence in $\mathcal{L}_2(\overline{Q})$ where $\overline{Q}$ arbitrary finite R subregion. Function $\overline{\mathcal{U}}(g, K)$ is the solution of equation

$$P[\overline{\mathcal{U}}] \equiv \left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} + K^2\right] \overline{\mathcal{U}}(g) = f(g) + K^2 \left[1 - \varepsilon(g)\right] \overline{\mathcal{U}}(g) \tag{6}$$

The operator P is defined for $\mathcal{L}_2(R)$ functions satisfying boundary conditions (2).

<u>Lemma 1</u>. The solution of (6) exists, is unique and may be presented as

$$\overline{\mathcal{U}}(g, K) = -\int_Q G(g, g_0, K, \Phi)\, f(g_0)\, dg_0\,; \; 0 < \arg K < \pi/2 \tag{7}$$

that is it coincides with the solution (1),(2), singling out by radiation principle for corresponding values of k. The proof of Lemma 1 is proceeding from the following statement: the solution (1),(2) in form (7) satisfies (6) and is in $\mathcal{L}_2(R)$ (in region $|\arg K| < \pi/2$ $\mathrm{Jm}\,\Gamma_n(K) > 0$ for all $n=0,\pm 1$ /2/, and hence, $\mathcal{U}(g)$ exponentially decreases when $|z|$ increases). Passing to the limit in (7) $\mathrm{Jm}K \to 0$, and taking into account the continuity of convolution and properties of

$G(g, g_0, \kappa, \Phi)$, we conclude that the following theorem is valid.

Theorem 2 (limitary attenuation principle). If $\mathrm{Re}\,\kappa \notin \{\overline{\kappa}_n\}$, then the limit (6) exists and singles out the unique solution (1),(2), satisfying radiation conditions (3). The limitary attenuation principle is equivalent to radiation principle while they single out the same solution of (1),(2) on the same set of values k (see theorem 1). We have to remark that all restrictions of the considered principles possible application areas may be reformulated in the terms of problem (6) eigen values and eigen functions so far as the nontrivial solution of (1)-(3) in the set $\{\overline{\kappa}_n\} \in \Omega$ points, not coinciding with branching points κ_n, according to unique solution theorem from /2/, are belonging to $\mathcal{L}_2(R)$.

4. On quasi-periodic grating excitation the limitary amplitude principle consists in that solution of (1),(2) may be presented as limit (convergence in $\mathcal{L}_2(\overline{Q})$)

$$\lim_{t \to \infty} v(g,t)\, e^{i\omega t} = u(g), \quad \mathrm{Im}\,\omega = 0, \tag{8}$$

where $v(g,t)$ - the solution of wave equation

$$\left[-\varepsilon(g)\varepsilon_0\mu_0 \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right] v(g,t) = f(g)\, e^{-i\omega t}, \quad t > 0 \tag{9}$$

satisfying conditions (2) with $t > 0$ and zero initial dates with $t=0$.

Theorem 3 (limitary amplitude principle). Let $\kappa = \omega/c$, $c = (\varepsilon_0\mu_0)^{-1/2}$ is not branching point and Green function $G(g, g_0, \kappa, \Phi)$ has no valid poles on the k plane (second demand is always fulfilled when $\mathrm{Im}\,\varepsilon(g) \neq 0$). Then the limit (8) singles out the unique solution of (1),(2). This solution satisfies radiation condition (3). If the second demand is not fulfilled, but point $\kappa \notin \{\overline{\kappa}_n\}$ and source function $f(g)$ is such, that integral (4) turns to zero for all $\overline{\kappa} \in \{\overline{\kappa}_n\}$, then (8) is true. In contraries the limitary amplitude principle as the single out of unique boundary value problem solution instrument does not work (in not valid). It is clear that under definite conditions the limitary amplitude principle is equivalent with radiation one and limitary attennuation principle, but in common case the range of its possible application is much more narrow.

5. Let us consider the grating excited by local source $f(g)$. Without any community restriction, one may consider the bearer $f(g)$ is fully in Q.

Theorem 4 (local excitation radiation principle). If the grating has no eigen valid waves, transfering energy along the direction of periodicity y with given $k > 0$, then radiation condition

$$u(g,\kappa) = \frac{e^{i\kappa\tau}}{\tau^{1/2}} A(\kappa,\varphi) + O(\tau^{-1}) + \begin{cases} \sum\limits_{\Phi_\kappa \in M^+} b_\kappa^+\, u(g,\Phi_\kappa), & y \to +\infty \\ \sum\limits_{\Phi_\kappa \in M^-} b_\kappa^-\, u(g,\Phi_\kappa), & y \to -\infty \end{cases} ; \ \tau \to \infty \tag{10}$$

singles out the unique solution of (1),(2), satisfying physically justified requirement that among partial components of the radiation field there should not be waves coming from infinity.

The proof of the existance of the solution in expression (10) comes to the determination of the "true" contour of integration in integral

$$\int_{-0,5}^{0,5} G(g,g_0,\Phi)\,d\Phi\ ,\quad g_0 \in Q,$$

that gives the formal representation Green function of the grating in the point source g_0 field. The main difficulties are caused by the requirement to take into account zeros and poles of Green function $G(g,g_0,\Phi)$ in the quasi-periodic point sources field as the function of a complex variable Φ. The residues $G(g,g_0,\Phi)$ in the poles Φ_κ define gratings eigen waves $\mathcal{U}(g,\Phi_\kappa)$. The field $\mathcal{U}(g,\Phi_\kappa)$ exponentially decreases when $|z|$ increases if Φ_κ are real (real eigen waves). This fact gives the possibility to connect the quantity $P(\mathcal{U},\vec{y})$, defining the value and transmitter energy direction via complete ($|x| \leqslant 0.5$) cross section of transmitting structure by plane y=const ($\operatorname{Re} P(\mathcal{U},\vec{y}$) is independent from y). M^+ and M^- finite set of real eigennumbers Φ_κ , connected with eigenmodes transfering the energy along the grating in the positive and negative directions of the y axis. The proof of the unique can be done similar to /2/ with the help of complex power theorem analogy, applied to grating eigen modes.

References

1. Shestopalov V.P., Litvinenko L.N., Masalov S.A., Sologub V.G. Wave diffraction on gratings. Kharkov: Univ. Press, 1973, 287p.
2. Sirenko Yu.K., Shestopalov V.P. DAN, 1985, v.285, N 2, p.335-338.
3. Sirenko Yu.K., Shestopalov V.P. DAN, 1986, v.286, N 1, p.85-88.