# MUTUAL IMPEDANCE BETWEEN WIRE ANTENNAS IN A CIRCULAR WAVEGUIDE

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## I. INTRODUCTION

The mutual impedance between linear antennas in free space has been studied by many researchers, but this problem in a waveguide was little paid attention to. The mutual impedance between probes, vertically located on the broad wall of a rectangular waveguide is derived by Ittipiboon and Shafai [1] using the vector potential $\vec{A}$, and arbitrarily located in a rectangular waveguide was analyzed recently by the author [2] using the DGF $\overline{G}$. The investigations are extremely useful in design of antennas with specific uses, microwave circuit and various filters.

However, to my knowledge, the mutual impedance for a circular waveguide has not been considered. In this paper, the field distribution and mutual coupling of dipole antenna in circular waveguide have been studied in detail. The general formulas of mutual impedance between probes are given. In derivation, the DGF, field transformation, and reaction theorem are used. The waveguide is semi-infinite. The reflection coefficient at the terminal plane is $\Gamma$. The lengths, feeding points, and orientations of the two antennas in the waveguide are all arbitrary. This method is general and can be used to solve similar problems in waveguides of different cross sections and in cavity resonators.

## II. THE DYADIC GREEN'S FUNCTION

The problem to be considered is shown in Fig. 1. Two dipole antennas, arbitrarily orientated, are located in a circular waveguide. Suppose the radius of the waveguide is a and is filled with air ($\mu_0, \varepsilon_0$). The DGF $\overline{G}$ of the first kind pertaining to the waveguide is given by

$$
\begin{aligned}
\overline{G}(\vec{r},\vec{r}') = & -\frac{1}{k^2}\delta(\vec{r}-\vec{r}')\,\hat{z}\hat{z} + \frac{j}{4\pi k^2}\sum_n\sum_m (2-\delta_0) \\
& \cdot\Big\{ \hat{\rho}\hat{\rho}'\left[n^2 k_1 M_\rho M'_\rho (e_{1\mu}+\Gamma e_{2\mu}) + \lambda^2 k_2 N_\rho N'_\rho (e_{1\lambda}+\Gamma e_{2\lambda})\right] \\
& + \hat{\rho}\hat{\phi}'\left[n\mu k_1 M_\rho M'_\phi (e_{1\mu}+\Gamma e_{2\mu}) - n\lambda k_2 N_\rho N'_\phi (e_{1\lambda}+\Gamma e_{2\lambda})\right] \\
& + \hat{\rho}\hat{z}\left[\lambda k_3 N_\rho N'_z (\pm e_{1\lambda} - \Gamma e_{2\lambda})\right. \\
& + \hat{\phi}\hat{\rho}'\left[n\mu k_1 M_\phi M'_\rho (e_{1\mu}+\Gamma e_{2\mu}) - n\lambda k_2 N_\phi N'_\rho (e_{1\lambda}+\Gamma e_{2\lambda})\right] \\
& + \hat{\phi}\hat{\phi}'\left[\mu^2 k_1 M_\phi M'_\phi (e_{1\mu}+\Gamma e_{2\mu}) + n^2 k_2 N_\phi N'_\phi (e_{1\lambda}+\Gamma e_{2\lambda})\right] \\
& + \hat{\phi}\hat{z}\left[n k_3 N_\phi N'_z (\mp e_{1\lambda}+\Gamma e_{2\lambda})\right] + \hat{z}\hat{\rho}'\left[\lambda k_3 N_z N'_\rho (\mp e_{1\lambda} - \Gamma e_{2\lambda})\right] \\
& + \hat{z}\hat{\phi}'\left[n k_3 N_z N'_\phi (\pm e_{1\lambda}+\Gamma e_{2\lambda})\right] + \hat{z}\hat{z}\left[k_4 N_z N'_z (e_{1\lambda} - \Gamma e_{2\lambda})\right]\Big\},\quad z \gtrless z' \qquad (1)
\end{aligned}
$$

where

$$M_\rho = \frac{1}{\rho} J_n(\mu\rho)\sin(n\phi-\phi_p) \qquad M_\phi = J_n'(\mu\rho)\cos(n\phi-\phi_p)$$

$$N_\rho = J_n'(\lambda\rho)\cos(n\phi-\phi_p) \qquad N_\phi = \frac{1}{\rho} J_n(\lambda\rho)\sin(n\phi-\phi_p)$$

$$N_z = J_n(\lambda\rho)\cos(n\phi-\phi_p)$$

$$k_1 = \frac{k^2}{\mu^2 I_\mu k_\mu} \qquad k_2 = \frac{k_\lambda}{\lambda^2 I_\lambda} \qquad k_3 = \frac{j}{I_\lambda} \qquad K_4 = \frac{\lambda^2}{I_\lambda k_\lambda}$$

$$e_{1i} = e^{\pm jk_i(z-z')} \qquad e_{2i} = e^{jk_i(z+z')} \quad , \ i = \mu, \lambda$$

The independent variables of $M_\rho'$ , $M_\phi'$ , $N_\rho'$ , $N_\phi'$ and $N_z'$ are $\rho'$ and $\phi'$ .

## III. FIELD $\vec{E}_1$ RADIATED BY PROBE 1

Assume that the current distribution (flowing in the $\zeta$ - direction) of antenna 1 is given by

$$\vec{J}_1(\zeta) = \hat{\zeta} I_1 \delta(\xi)\delta(\eta) \tag{2a}$$

$$I_1 = \begin{cases} I_{10} \dfrac{\sin k(\zeta-\zeta_1)}{\sin k(\zeta_2-\zeta_1)} , & \zeta_1 \le \zeta \le \zeta_2 \\[2ex] I_{10} \dfrac{\sin k(\zeta_3-\zeta)}{\sin k(\zeta_3-\zeta_2)} , & \zeta_2 \le \zeta \le \zeta_3 \end{cases} \tag{2b}$$

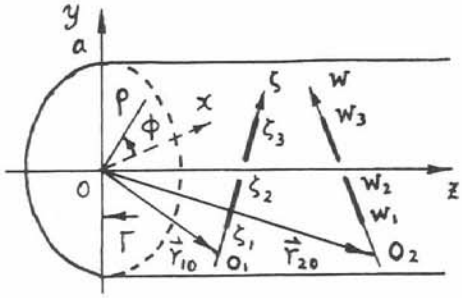


Fig. 1

where $I_{10}$ is feeding point current, k is the free space wave number. The electric field radiated from antenna 1 is

$$\vec{E}_1 = j\omega\mu_0 \int_{v'} \bar{G}\cdot\vec{J}_1 \, dv' = j\omega\mu_0 \int_{\zeta_1}^{\zeta_3} \bar{G}\cdot I_1 \, d\vec{\zeta} \tag{3}$$

where $d\vec{\zeta} = \hat{x}dx' + \hat{y}dy' + \hat{z}dz'$. Assume that the direction cosines of $\hat{\zeta}$ are $S_1 = \cos\alpha_1$ , $S_2 = \cos\beta_1$ , $S_3 = \cos\gamma_1$ . Hence, the parametric equations of $\zeta$ axis are given by

$$x' = x_\zeta = S_1\zeta + x_{10} \qquad y' = y_\zeta = S_2\zeta + y_{10} \qquad z' = z_\zeta = S_3\zeta + z_{10} \tag{4}$$

Through tedious treatment of (3), the principal value of the integral for $z > z'$ and $z < z'$ are found to be, respectively

$$\begin{pmatrix} E_\rho \\ E_\phi \\ E_z' \end{pmatrix} = \sum_n \sum_m (\phi_{ij}) \begin{pmatrix} A_1 M_x F_\mu(z) + A_2 N_x F_\lambda(z) \\ A_1 M_y F_\mu(z) + A_2 N_y F_\lambda(z) \\ 0 \qquad + A_3 N_z F_\lambda(z) \end{pmatrix} , \qquad z > z' \tag{5a}$$

$$\begin{pmatrix} E_\rho \\ E_\phi \\ E_z' \end{pmatrix} = \sum_n \sum_m (\phi_{ij}) \begin{pmatrix} B_1 M_x F_\mu(+\Gamma z) + B_2 N_x F_\lambda(+\Gamma z) \\ B_1 M_y F_\mu(+\Gamma z) + B_2 N_y F_\lambda(+\Gamma z) \\ 0 \qquad + B_3 N_z F_\lambda(-\Gamma z) \end{pmatrix} , \quad z < z' \tag{5b}$$

In (5), a correction term $E_z''$ should be added to the term $E_z'$ to yield a correct longitudinal component $E_z$, that is, $E_z = E_z' + E_z''$ . Based on the distribution theory, the correction term is given by

$$E''_z(\rho,\phi,z) = -\frac{j\eta_0}{2k\pi}\sum_n\sum_m \frac{2-\delta_0}{I_\lambda}\frac{S_3}{|S_3|} I_1(\zeta) J_n(\lambda\rho_\zeta) \times J_n(\lambda\rho)\cos n(\phi-\phi_\zeta)\Big|_{\zeta=(z-z_{10})/S_3} \tag{5c}$$

where

$$\begin{cases} M_x = n M_\rho \cos\phi - \mu M_\phi \sin\phi & M_y = n M_\rho \sin\phi + \mu M_\phi \cos\phi \\ N_x = \lambda N_\rho \cos\phi + n N_\phi \sin\phi & N_y = \lambda N_\rho \sin\phi - n N_\phi \cos\phi \\ F_i(\pm\Gamma z) = e^{-jk_i z} \pm \Gamma e^{jk_i z} & F_i(z) = e^{jk_i z}, \quad i=\mu,\lambda \\ \phi_{11} = \phi_{22} = \cos\phi \quad \phi_{12} = -\phi_{21} = \sin\phi & \phi_{33} = 1 \quad \text{other} \\ \rho_\zeta = (x_\zeta^2 + y_\zeta^2)^{1/2} & tg\,\phi_\zeta = y_\zeta / x_\zeta \end{cases} \tag{6}$$

The costants A and B are given by

$$\begin{cases} A_1 = R_0(S_1 k_1 P_{Mx} + S_2 k_1 P_{My}) \qquad A_2 = R_0(S_1 k_2 P_{Nx} + S_2 k_2 P_{Ny} + S_3 k_3 P_{Nz}) \\ A_3 = R_0(-S_1 k_3 P_{Nx} - S_2 k_3 P_{Ny} + S_3 k_4 P_{Nz}) \end{cases} \tag{7a}$$

$$\begin{cases} B_1 = R_0(S_1 k_1 Q_{Mx} + S_2 k_1 Q_{My}) \qquad B_2 = R_0(S_1 k_2 Q_{Nx} + S_2 k_2 Q_{Ny} - S_3 k_3 Q_{Nz}) \\ B_3 = R_0(S_1 k_3 Q_{Nx} + S_2 k_3 Q_{Ny} + S_3 k_4 Q_{Nz}) \qquad R_0 = -\eta_0(2-\delta_0)/(4\pi) \end{cases} \tag{7b}$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the intrinsic impedance of free space, and

$$\begin{cases} P_{Mi} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1 m_i(\zeta) f_\mu(+\Gamma\zeta)\,d\zeta \qquad Q_{Mi} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1 m_i(\zeta) f_\mu(\zeta)\,d\zeta \\ P_{Ni} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1 n_i(\zeta) f_\lambda(+\Gamma\zeta)\,d\zeta \quad , \quad i = x, y \\ P_{Nz} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1 n_z(\zeta) f_\lambda(-\Gamma\zeta)\,d\zeta \qquad Q_{Ni} = \frac{1}{k}\int_{\zeta_1}^{\zeta_3} I_1 n_i(\zeta) f_\lambda(\zeta)\,d\zeta, \quad i = x, y, z \end{cases} \tag{8}$$

where

$$\begin{cases} m_x(\zeta) = \frac{n x_\zeta}{\rho_\zeta^2} J_n(\mu\rho_\zeta)\sin(n\phi_\zeta - \phi_p) - \frac{\mu y_\zeta}{\rho_\zeta} J'_n(\mu\rho_\zeta)\cos(n\phi_\zeta - \phi_p) \\ m_y(\zeta) = \frac{n y_\zeta}{\rho_\zeta^2} J_n(\mu\rho_\zeta)\sin(n\phi_\zeta - \phi_p) + \frac{\mu x_\zeta}{\rho_\zeta} J'_n(\mu\rho_\zeta)\cos(n\phi_\zeta - \phi_p) \\ n_x(\zeta) = \frac{\lambda x_\zeta}{\rho_\zeta} J'_n(\lambda\rho_\zeta)\cos(n\phi_\zeta - \phi_p) + \frac{n y_\zeta}{\rho_\zeta^2} J_n(\lambda\rho_\zeta)\sin(n\phi_\zeta - \phi_p) \\ n_y(\zeta) = \frac{\lambda y_\zeta}{\rho_\zeta} J'_n(\lambda\rho_\zeta)\cos(n\phi_\zeta - \phi_p) - \frac{n x_\zeta}{\rho_\zeta^2} J_n(\lambda\rho_\zeta)\sin(n\phi_\zeta - \phi_p) \\ n_z(\zeta) = J_n(\lambda\rho_\zeta)\cos(n\phi_\zeta - \phi_p) \\ \qquad f_i(\pm\Gamma\zeta) = e^{-jk_i z_\zeta} \pm \Gamma e^{jk_i z_\zeta}, \quad i = \mu, \lambda \\ \qquad f_i(\zeta) = e^{jk_i z_\zeta}, \qquad i = \mu, \lambda \end{cases} \tag{9}$$

## IV. MUTUAL IMPEDANCE

In order to calculate mutual impedance, we must determine the tangential component $E_{1w}$ of $\vec{E}_1$ along antenna 2. The current distribution of antenna 2 is similar to that of antenna 1, except that $I_1 \to I_2$, $\zeta \to w$. Assume that the direction cosines of $\hat{w}$ are $t_1 = \cos\alpha_2$, $t_2 = \cos\beta_2$, $t_3 = \cos\gamma_2$. The parametric equations of w axis are given by

$$x_w = t_1 w + x_{20} \qquad y_w = t_2 w + y_{20} \qquad z_w = t_3 w + z_{20} \tag{10}$$

The tangential component $E_{1w}$ is given by

$$E_{1w} = \vec{E}_1 \cdot \hat{w} = (t_1 \cos\phi + t_2 \sin\phi) E_\rho + (-t_1 \sin\phi + t_2 \cos\phi) E_\phi + t_3 E_z \tag{11}$$

Substituting for $\vec{E}_1$ from (5), we get for $z > z'$ and $z < z'$ respectively

$$E_{1w} = \sum_n \sum_m \Big\{ t_1 \left[ A_1 m_x(w) f_\mu(w) + A_2 n_x(w) f_\lambda(w) \right] + t_2 \left[ A_1 m_y(w) f_\mu(w) + A_2 n_y(w) f_\lambda(w) \right] + t_3 \left[ A_3 n_z(w) f_\lambda(w) + E_z''(\rho_w, \phi_w, z_w) \right] \Big\}, \quad z > z' \tag{12a}$$

$$E_{1w} = \sum_n \sum_m \Big\{ t_1 \left[ B_1 m_x(w) f_\mu(+\Gamma w) + B_2 n_x(w) f_\lambda(+\Gamma w) \right] + t_2 \left[ B_1 m_y(w) f_\mu(+\Gamma w) + B_2 n_y(w) f_\lambda(+\Gamma w) \right] + t_3 \left[ B_3 n_z(w) f_\lambda(-\Gamma w) + E_z''(\rho_w, \phi_w, z_w) \right] \Big\}, \quad z < z' \tag{12b}$$

where $m_i(w)$, $n_i(w)$, $f_i(\pm\Gamma w)$ and $f_i(w)$ are similar to that of (9), except that $\zeta \to w$. And

$$\rho_w = (x_w^2 + y_w^2)^{1/2} \qquad tg\,\phi_w = y_w / x_w \tag{13}$$

By the reaction concept, the mutual impedance between the two antennas is given by

$$M = -\frac{1}{I_{10} I_{20}} \int_{w_1}^{w_3} E_{1w} I_2 \, dw \tag{14}$$

## REFERENCES


1. A. Ittipiboon and L. Shafai, IEEE Trans., MTT - 33, pp. 327 - 335, Apr. 1985.

2. B.S. Wang, IEEE Trans., MTT - 36, pp. 53 - 60, Jan. 1988.