THE CORRELATIVE POTENTIAL FUNCTION AND ITS APPLICATION IN ELECTROMAGNETIC WAVE THEORY

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Historically, Helmholtz's method was used to find the field $\vec{f}$ satisfying the set of equations $\nabla\times\vec{f}=\vec{\omega}$, $\nabla\cdot\vec{f}=P$. In this paper, a new method is deduced in terms of the transformation between the curl function and the divergence function through introducing a new potential function ψ, called correlative potential function. The procedures of solving the equations are facilitated. We may only solve one poisson equation and one Neumann problem in order to find $\vec{f}$.

If this method is applied to Maxwell equations, we solve neither four wave equations for the retarded potentials $\vec{A}$ and ϕ (Certainly, we do not need to solve ϕ in terms of the Lorentz condition), nor three wave equations for the Hertzian vector potential $\vec{\pi}$. We need only to solve two wave equations for the electric-type retarded correlative potential ψ_e and the magnetic -type retarded correlative potential ψ_m.

In terms of this method, the procedures for finding solutions of the vector fields are simplified.

Consider the set of equations and its solution for the general vector field. In a volume V and on a closed surface S surrounding the volume V, a vector field $\vec{f}$ satisfies the following set of equations

$$\nabla\times\vec{f}=\vec{\omega} \quad (\text{ in V }) \tag{1.1a}$$
$$\nabla\cdot\vec{f}=P \quad (\text{ in V }) \tag{1.1b}$$
$$\vec{n}\cdot\vec{f}=g(M) \quad (\text{ on S }) \tag{1.1c}$$

$$\vec{\omega}=\vec{e}_1\omega_1+\vec{e}_2\omega_2+\vec{e}_3\omega_3 \tag{1.2}$$

where $\vec{\omega}$ and P are qiven function, $\vec{n}$ is the unit vector outward normal to the surfaces S, M is a point on S as shown in Fig.1.1. And g(M) and P satisfy the relation

$$\oint_S g(M)ds=\int_V Pdv \tag{1.3}$$

From eq. (1.1a), we have

$$\nabla\cdot(\nabla\times f)=\nabla\cdot\vec{\omega}=0 \tag{1.4}$$

According to Helmholtz's theorem, the solution of the set of eqs. (1.1) is unique.

We have proved that the solution of eqs. (1.1) is given by

$$f=-\nabla\psi+\vec{e}_2\left(\frac{1}{h_2}\int h_1 h_2\omega_3 du_1\right)+\vec{e}_3\left(-\frac{1}{h_3}\int h_3 h_1\omega_2 du_1\right)-\nabla\psi'=\vec{F}+\vec{F}' \tag{1.5}$$

where

$$\vec{F}=\vec{e}_1\left(-\frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\right)+\vec{e}_2\left(\frac{1}{h_2}\int h_1 h_2\omega_3 du_1-\frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\right)$$
$$+\vec{e}_3\left(-\frac{1}{h_3}\int h_3 h_1\omega_2 du_1-\frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\right)$$
$$=-\nabla\psi+\vec{e}_2\left(\frac{1}{h_2}\int h_1 h_2\omega_3 du_1\right)+\vec{e}_3\left(-\frac{1}{h_3}\int h_3 h_1\omega_2 du_1\right) \qquad (1.6)$$

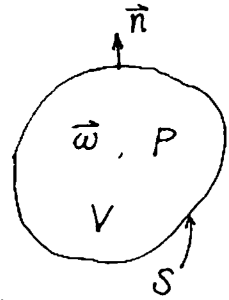

Fig. 1.1

The function ψ satisfies the Poisson's equation

$$\Delta\psi=-P+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\int h_1 h_2\omega_3 du_1\right)-\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\int h_3 h_1\omega_2 du_1\right)\right] \qquad (1.7)$$

ψ is called the correlative potential of the vector field. And

$$F'=-\nabla\psi' \qquad (1.8)$$

The function ψ' satisfies Laplace's equation and the boundary condition.

$$\nabla\psi'=0 \qquad (\text{ in V })$$

$$\vec{n}\cdot\vec{F}'=-\frac{\partial\psi'}{\partial n}=\vec{n}\cdot\vec{f}-\vec{n}\cdot\vec{F}=g(M)-\vec{n}\cdot\vec{F} \qquad (\text{ on S }) \qquad (1.9)$$

where u_1, u_2, u_3 are the variables of the orthogonal curvilinear coordinates, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are unit vectors and h_1, h_2, h_3 are scale factors.

To illustrate the use of (1.7), let us discuss some examples.

In a vacuum (μ_0) the z-directed current density distributions are given by

$$J=\begin{cases}\vec{K}J_0\, r & (\ r\leqslant R\ )\\ 0 & (\ r>R\ )\end{cases}$$

as shown in Fig.1.2. Find the magnetic fields inside the cylinder (r<R) and outside the cylinder (r>R).

Using the above new formula,we can get the magnetic fields inside and outside the cylinder respectively

$$\vec{H}_1=\vec{e}_\phi\frac{1}{3}J_0 r^2 \qquad (1.29)$$

$$\vec{H}_2=\vec{e}_\phi\frac{1}{3}J_0 R^3\frac{1}{r} \qquad (1.30)$$

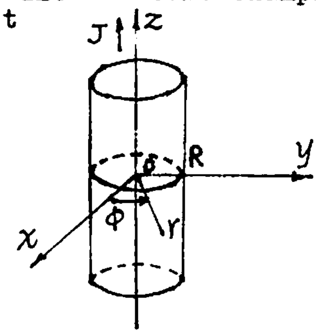

Fig. 1·2

These results are identical with those by the classical method.

Next, assume that the impressed current density $\vec{J}=\vec{i}J_x+\vec{j}J_y+\vec{k}J_z$ and charge density ρ are given. In a vacuum, the classical retarded potentials $\vec{A}$, ϕ and the fields $\vec{E}$, $\vec{H}$ satisfy

$$\Delta\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J} \qquad (2.1a)$$

$$\Delta\phi-\mu_0\varepsilon_0\frac{\partial^2\phi}{\partial t^2}=-\frac{\rho}{\varepsilon_0} \qquad (2.1b)$$

$$\vec{E}=-\frac{\partial\vec{A}}{\partial t}-\nabla\phi \qquad (2.1c)$$

$$\vec{H}=\frac{1}{\mu_0}\nabla\times A \qquad (2.1d)$$

In order to find $\vec{E}$ and $\vec{H}$, three wave equations must be solved for $\vec{A}$ at least.

We have shown that if the new method is used, only two scalar wave equations of ψ_e and ψ_m are sufficient for obtaining $\vec{E}$ and $\vec{H}$, namely,

$$\Delta\psi_e-\mu_0\varepsilon_0\frac{\partial^2\psi_e}{\partial t^2}=-\frac{\rho}{\varepsilon_0}-\mu_0\int\frac{\partial J_x}{\partial t}dx \tag{2.2}$$

$$\Delta\psi_m-\mu_0\varepsilon_0\frac{\partial^2\psi_m}{\partial t^2}=\int\left(\frac{\partial J_y}{\partial x}-\frac{\partial J_x}{\partial y}\right)dz \tag{2.3}$$

$$\vec{E}=-\nabla\psi_e+j\mu_0\int\frac{\partial^2\psi_m}{\partial z\partial t}dx$$

$$+\vec{k}\left[-\mu_0\iint\frac{\partial J_x}{\partial t}dz\,dx+\mu_0\varepsilon_0\int\frac{\partial^2\psi_e}{\partial t^2}dz-\mu_0\int\frac{\partial^2\psi_m}{\partial y\partial t}dx\right] \tag{2.4}$$

$$\vec{H}=-\nabla\psi_m+\vec{i}\left[\int J_y\,dz+\mu_0\varepsilon_0\int\frac{\partial^2\psi_m}{\partial t^2}dx-\varepsilon_0\int\frac{\partial^2\psi_e}{\partial y\partial t}dz\right]$$

$$+\vec{j}\left[-\int J_x\,dz+\varepsilon_0\int\frac{\partial^2\psi_e}{\partial x\partial t}dz\right] \tag{2.5}$$

In the above equations the subscripts x, y and z may be cycled by the right-hand rule.

To illustrate the use of (2.2)-(2.5), we now discuss the fields of Hertzian dipole.

The Hertzian dipole is a linear current element $I=I_0\exp(j\omega t)$ (ω—angle frequency) of length l, oriented in the z -direction and located at the origin as shown in Fig.2.1. For convenience we assume that I_0 is a real amplitude factor. The charges $\pm q$, $q=\frac{1}{j\omega}I_0$ are at the upper and lower end of the current element respectively.

Using the new formula, we have obtained the following results

$$H_r=0$$

$$H_\theta=0$$

$$H_\phi=-\sin\phi H_x+\cos\phi H_y$$

$$=j\frac{\omega q l}{4\pi}\sin\theta\left(\frac{1}{r^2}+\frac{jK_0}{r}\right)\exp(-jK_0r)$$

$$=\frac{I_0 l}{4\pi}\sin\theta\left(\frac{jK_0}{r}+\frac{1}{r^2}\right)\exp(-jK_0r) \tag{2.26}$$

$$\vec{E}=-j\frac{I_0 l}{2\pi\omega\varepsilon_0}\cos\theta\left(\frac{jK_0}{r^2}+\frac{1}{r^3}\right)\exp(-jK_0r)\vec{e}_r$$

$$-j\frac{I_0 l}{4\pi\omega\varepsilon_0}\sin\theta\left(-\frac{K_0^2}{r}+\frac{jK_0}{r^2}+\frac{1}{r^3}\right)\exp(-jK_0r)\vec{e}_\theta \tag{2.28}$$

The results are identical with those by the classical methods.

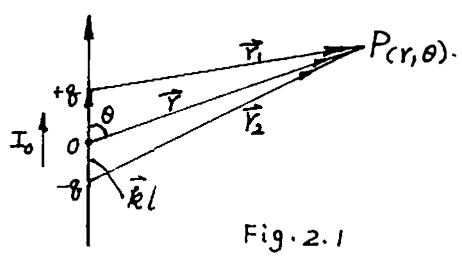

Fig.2.1

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