THE APPROXIMATION OF THE PARABOLIC EQUATION FOR WEAK AND STRONG FLUCTUATIONS OF THE POINT SOURCE FIELD IN THE INHOMOGENEOUS MEDIA WITH THE RANDOM COMPONENT

Nikolay N. Zernov
Institute of Physics. University of St.Petersburg.
Ulyanovskaya 1. Petrodvorets. 198904 St.Petersburg. Russia

The submitted paper deals with the parabolic equations and their solutions for the partial waves of the integral representations of the point source field in the inhomogeneous media with fluctuations of dielectric permeability. The weak and strong fluctuations of the field are considered. The attention is emphasized at the case, when the undisturbed incident partial waves have their own caustics, for which the parabolic equation of rather general form is constructed and solved. That is important for HF wave propagation in the ionosphere with fluctuations of the electron density. In the case of weak fluctuations the solution is constructed for a separate realization of the point source field. For strong fluctuations of the amplitude the point source mean field is considered by the formalism of the parabolic equation for partial waves in the integral representation of the mean field.

First we consider the integral representation (N.N.Zernov, Sov. Radiotehnika i elektronika, 1990 ,v.35 ,N8 ,p.1590 in Russian; N.N.Zernov et al. Radio Science Spec. Issue of JS2 for the 23-d Gen. Assembly of URSI, Prague, 1990, to be published)

$$U(\vec{r}) = \frac{i}{4\pi}\int_{-\infty}^{+\infty} v_0(\vec{r},\alpha)\exp[\varphi(\vec{r},\alpha)]\,d\alpha, \quad \vec{r} = \{x, z\}, \tag{1}$$

which generalizes Rytov method for the case of a point source field in the inhomogeneous media, being the asymptotic solution of the equation

$$\nabla^2 U + K^2[\varepsilon_0(z) + \varepsilon(\vec{r})]U = \delta(\vec{r}) \tag{2}$$

with the plane-stratified media $\varepsilon_0(z)$, modeling the undisturbed ionosphere, the local inhomogeneities $\varepsilon(\vec{r})$ and the wave number K for vacuum. The function $v_0(\vec{r},\alpha)$ is the integrand of Fourier integral for the point source field in the plane-stratified media, satisfying the reduced equation (2) with $\varepsilon(\vec{r}) = 0$. The function $\varphi(\vec{r},\alpha)$ describes the diffraction of the partial waves on the local inhomogeneities. The first and second order terms of the perturbation series for φ, when $\varepsilon(\vec{r})$ is a weak disturbance, satisfy the equations

$$2\left(\frac{\nabla v_0}{v_0}, \varphi_m\right) + \nabla^2\varphi_m = f_m \tag{3}$$

with m=1,2, $f_1 = -K^2\varepsilon(\vec{r})$, $f_2 = -(\nabla\varphi_1)^2$ and the uniform boundary conditions $\varphi_m \to 0$ with respect to α when $r \to 0$, that is quite really for forward scattering inhomogeneities. The partial wave $v_0(\vec{r},\alpha)$ in (1) one can express in WKB approximation

$$v_0(\vec{r},\alpha) = [\varepsilon_0(0) - \alpha^2]^{-1/4}[\varepsilon_0(z) - \alpha^2]^{-1/4} \times \tag{4}$$

$\exp[ik\alpha x + ikL(z,\alpha)]$

outside of caustic, with $L(z,\alpha) = \int_0^z [\varepsilon_0(z) - \alpha^2]^{1/2} dz$

before the reflection from caustic $\tilde{z}$, determined by the equality $\varepsilon_0(\tilde{z}) - \alpha^2 = 0$, or if there is no caustic at all; and

$$L(z,\alpha) = \int_0^{\tilde{z}} [\varepsilon_0(z) - \alpha^2]^{1/2} dz + \int_z^{\tilde{z}} [\varepsilon_0(z) - \alpha^2]^{1/2} dz$$

after the reflection from caustic.

Near the turning point $\tilde{z}$ the field $v_0(\vec{r},\alpha)$ must be expressed by that of Airy functions, which gives the standing wave below caustic and exponential descent after it. For further consideration we must only know, that the near caustic area scale is

$$\ell_c \sim K^{-2/3} \left| \frac{d\varepsilon_0}{dz}(\tilde{z}) \right|^{-1/3}. \quad (5)$$

As in (N.N.Zernov. Sov. Trudi . XIY Vsesouzn. Konf. po Rasprostr. Radio voln. L. Nauka. 1984, t.2, str.244; in Russian) for each and point of observation (x,z) we choose, speaking by the words of (R.J.Hill. Journ. Acoust. Soc. Amer. 1985, v.77, Nr.5, p.1742), the particular ray of interest, which comes at the point (x,z) and arises from some point on the constant phase surface, containing the point source and propagating through the source with the angle $\theta = \arcsin \alpha/\sqrt{\varepsilon_0(0)}$ with the Z-axis. We use this particular ray to introduce around it the local ray-centered coordinate system (s,n) with the variable s to be the distance along the ray, and n - the distance between the point (x,z) and the particular ray along the straight line perpendicular to this ray. For the coordinates chosen the Lame coefficients are the following

$$h_s = 1 + n/\rho(s), \quad h_n = 1$$

with $\rho(s)$ to be the curvature radius of the ray of interest.

Taking into account the expressions for scalar product of the gradients and Laplacian for these curved coordinates, one can write instead of (3) the equation

$$\frac{2}{v_0}\frac{\partial v_0}{\partial s}\frac{\partial \varphi_m}{\partial s} + h_s^2\left[\frac{\partial^2 \varphi_m}{\partial n^2} + \frac{2}{v_0}\frac{\partial v_0}{\partial n}\frac{\partial \varphi_m}{\partial n}\right] + \frac{\partial^2 \varphi_m}{\partial s^2} - h_s^{-1}\frac{\partial h_s}{\partial s}\frac{\partial \varphi_m}{\partial s} - \frac{h_s}{\rho}\frac{\partial \varphi_m}{\partial n} = h_s^2 f_m.$$

This equation is of the same sort as the equation in (R.J.Hill,1985), written for the ratio v/v_0. Under the conditions $K\ell_\varepsilon \gg 1$, $\ell_\varepsilon \rho^{-1} \ll 1$, $\ell_\varepsilon \rho^{-1} K^{-1} D^{-1} < 1$, where ℓ_ε is the spatial scale of the local inhomogeneities, D is the hop-distance, the last three terms of the equation can be shown to be small, compared with the first three. Therefore, the ultimate equation to be investigated further is the parabolic equation of the form

$$\frac{2}{v_0}\frac{\partial v_0}{\partial s}\frac{\partial \varphi_m}{\partial s} + h_s^2\left[\frac{\partial^2 \varphi_m}{\partial n^2} + \frac{2}{v_0}\frac{\partial v_0}{\partial n}\frac{\partial \varphi_m}{\partial n}\right] = h_s^2 f_m. \quad (6)$$

To construct the solution of the last equation we consider two areas of space. Out of caustic the field $v_0(s,n)$ is represented by formulae (4). After expanding the phase $\varphi_0(s,n)$ of $v_0(s,n)$ into the series near the point (s,0) one can write

$$\varphi_0(s,n) = K\int_0^s [\varepsilon_0(z(s,0))]^{1/2} ds - \frac{K\alpha n^2\sqrt{\varepsilon_0(s,0)}}{2\rho(s)\sqrt{\varepsilon_0(s,0)-\alpha^2}} + O(n^3).$$

Using the last expression, neglecting the derivatives of the slowly varying function $[\varepsilon_0(s,n)-\alpha^2]^{-1/4}$ and taking into account the conditions $\ell_\varepsilon/\rho \ll 1$ and $K\ell_\varepsilon^2\rho^{-1} \ll 1$, we get for the outside of the caustic area the equation

$$2iK\varepsilon_0^{1/2}(s,0)\frac{\partial\varphi_m}{\partial s} + \frac{\partial^2\varphi_m}{\partial n^2} = f_m. \tag{7}$$

In the vicinity of the turning point, where the n-axis is parallel to the z -axis, the second term in the equation (6) is small in comparison with the last one, if $\ell_c/\ell_\varepsilon < 1$, and the last term is of the order $K^{-1/3}|\varepsilon_0'(\tilde{z})|^{1/3} \ll 1$ with respect to the first. Then for the area near caustic we get the equation

$$2iK\varepsilon_0^{1/2}(s,0)\frac{\partial\varphi_m}{\partial s} = f_m. \tag{8}$$

So, in the case, when the incident field $v_0(s,n)$ has the simple caustic, the parabolic equation (6) for the complex phase of the scattered field has two representations (7) and (8) for two areas of space. In accordance with (8) the irregularities, having the spatial scales large compared with the caustic zone size, disturb only the phase of the scattered field at caustic.

For each of the equations (7), (8) it is easy to construct the solution, but these two solutions will not be the uniform solution of the initial equation (6) along the whole ray of interest. To get the uniform solution it is necessary to solve the equation (3) directly. It is done in the approximation of Fresnel diffraction for forward scattering (N.N.Zernov, 1990; N.N.Zernov et al. R. Sci., to be published). The solution is the following

$$\varphi_m(s(x,z),0) = 2^{-3/2}\pi^{-1/2}K^{-1/2}e^{-i\pi/4}\int f_m(x(s',n'),z(s',n'))\times \tag{9}$$

$$[\varepsilon_0(z(s',0))-\alpha^2]^{-1/2}M_{\alpha\alpha}^{-1/2}(z(s',0),\alpha)\exp\left\{-\frac{iK\varepsilon_0(s',0)M_{\alpha\alpha}^{-1}(s',0)}{2[\varepsilon_0(s',0)-\alpha^2]}n'^2\right\}ds'dn'.$$

Here $M_{\alpha\alpha}$ is the second derivative by α of the function $M(z,\alpha)$, adjusted with $L(z,\alpha)$ from formula (4). The Fresnel propagator in (9) has the infinitely fast oscillations near the points of the particular ray (s,0), where $\varepsilon_0(\tilde{z})-\alpha^2=0$, $M_{\alpha\alpha}(z_c,\alpha)=0$.

These are the caustics of the incident and secondary fields. At these points the expression (9) gives the finite geometrical optics solution of the equation like (8). It also can be shown, that out of caustics the equality (9) transits into the solution of the equation (7) with more simple Fresnel propagator, stipulated by (7).

Like that, the integral representation (1) with the complex phase (9), being the solution of the parabolic equation (6), gives the point source field in the inhomogeneous media with the weak irregularities, which can have stochastic as well as deterministic nature. The stochastic properties of the partial waves, such as level and phase fluctuations, and the moments of the whole field (1) can be described by that formalism developed.

The similar approach can be also used for the calculation of the mean field of the point source for the case, when the field amplitude fluctuations are strong. For this situation we write

$$\langle u(\vec{r})\rangle = \frac{i}{4\pi}\int_{-\infty}^{+\infty} v_0(\vec{r},\alpha)\langle w(\vec{r},\alpha)\rangle\, d\alpha \tag{10}$$

and we have for every realization of w the stochastic parabolic equation

$$2ik\varepsilon_0^{1/2}(s,0)\frac{\partial w}{\partial s} + h_s^2\left[\frac{\partial^2 w}{\partial n^2} + \frac{2}{v_0}\frac{\partial v_0}{\partial n}\frac{\partial w}{\partial n}\right] + h_s^2 k^2 \varepsilon(s,n)\, w = 0$$

with the initial condition $w(\vec{r},\alpha)\to 1$, when $r\to 0$, uniform with respect to α. Then in the same manner, as in (R.J.Hill, 1985), one can get for the mean value $\langle w\rangle$ the equation

$$2ik\varepsilon_0^{1/2}(s,0)\frac{\partial\langle w\rangle}{\partial s} + h_s^2\left[\frac{\partial^2\langle w\rangle}{\partial n^2} + \frac{2}{v_0}\frac{\partial v_0}{\partial n}\frac{\partial\langle w\rangle}{\partial n}\right] + \frac{ik^3 h_s^4}{4\varepsilon_0^{1/2}(s,0)} A_\varepsilon(n,n,s)\langle w\rangle = 0 \tag{11}$$

under the condition of Markov approximation

$$\langle \varepsilon(s,n)\,\varepsilon(s',n')\rangle = A_\varepsilon(n,n',s)\,\delta(s-s').$$

It is quite real to account, that the statistical inhomogeneity of the random function $\varepsilon(\vec{r})$ is due to the inhomogeneity of the undisturbed media $\varepsilon_0(z)$, and hence the function A_ε has the same spatial scale ℓ_0, as $\varepsilon_0(z)$. The equation (11) is written for the scale of n of the order ℓ_ε. That is why $A_\varepsilon(n,n,s)$ in (11) can be substituted by $A_\varepsilon(0,0,s)$ with the small mistake of the order $\ell_\varepsilon \ell_0^{-1}$. Then we finally have the equation

$$2ik\varepsilon_0^{1/2}(s,0)\frac{\partial\langle w\rangle}{\partial s} + \frac{\partial^2\langle w\rangle}{\partial n^2} + \frac{2}{v_0}\frac{\partial v_0}{\partial n}\frac{\partial\langle w\rangle}{\partial n} + \frac{ik^3 A_\varepsilon(0,0,s)}{4\varepsilon_0^{1/2}(s,0)}\langle w\rangle = 0,$$

($h_s=1$). Its solution is

$$\langle w\rangle = w_0(s,n)\exp\left[-\frac{k^2}{8}\int_0^S \frac{A_\varepsilon(0,0,s)}{\varepsilon_0(s,0)}\,ds\right] \tag{12}$$

with $w_0(s,n)$ to be the solution of the equation (11), when $A_\varepsilon = 0$. In our case $w_0 = 1$.

So, the integral (10) with v_0 from (4) and formula (12) for $\langle w\rangle$ give the integral representation of the mean field of the point source in the inhomogeneous media for the case of strong amplitude fluctuations. It describes the multi - ray effects of the mean field, including the mean field on caustics.