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MULTIPLE SCATTERING IN THE WAVE PROPAGATION THROUGH RANDOM MEDIA

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We consider the scalar wave equation

$$\{\nabla^2 + k^2[1+\epsilon\mu(\mathbf{r})]\}U(\mathbf{r}) = 0 \qquad (1)$$

which describes the propagation of waves through a random medium with refractive index $n(\mathbf{r}) = 1 + \epsilon\mu(\mathbf{r})$. Equation (1) may be solved by assuming $U = \Sigma_0^\infty \epsilon^j U_j$ (the Born method) or $U = \exp \Sigma_0^\infty \epsilon^j \psi_j$ (the Rytov method). Starting from the zero-th order solution assumed known, higher order terms of these methods are shown to be given successively as integrals involving lower order terms.[1]

On the other hand, a comparison of the forms of the solutions reveals a reciprocity relationship between the terms of the two methods.[2] Namely, by expanding $U = U_0 \exp \Sigma_1^\infty \epsilon^j \psi_j$ (with $U_0 = \exp \psi_0$) in powers of ϵ and comparing it with $U = U_0 \Sigma_0^\infty \epsilon^j \rho_j$, where $\rho_j = U_j/U_0$, we obtain

$$\rho_0 = 1;\ \rho_n = \sum_{j=1}^{n} \frac{1}{j!} \sum_{j_1+\cdots+j_j=n} \psi_{j_1}\psi_{j_2}\cdots\psi_{j_j},\ n=1,2,\cdots \qquad (2)$$

If we take the logarithm of $U = U_0(1+\Sigma_1^\infty \epsilon^j \rho_j)$, expand $\log(1+\Sigma_1^\infty \epsilon^j \rho_j)$ in powers of ϵ, and compare it with $\log U = \Sigma_0^\infty \epsilon^j \psi_j$ we obtain

$$\psi_0 = \log U_0;\ \psi_n = \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j} \sum_{j_1+\cdots+j_j=n} \rho_{j_1}\rho_{j_2}\cdots\rho_{j_j},\ n=1,2,\cdots \qquad (3)$$

In Eqs. (2) and (3), j_k take on non-zero positive integer values.

We can obtain analogue of Eq. (2) or (3) for truncated solutions $U^{(n)} = \exp \Sigma_0^n \epsilon^j \psi_j$ (the n<u>th</u> Rytov approximation) and ${}^{(n)}U = \Sigma_0^n \epsilon^j U_j$ (the n<u>th</u> Born approximation). Thus, $U^{(n)}$ may be represented by $U_0 \Sigma_0^\infty \epsilon^j \rho_j^{(n)}$, where

$$\rho_i^{(n)} = \left\{ \begin{array}{l} \sum_{j=1}^{i} \\ \sum_{j=[i/n]}^{i} \end{array} \right\} \frac{1}{j!} \sum_{j_1+\cdots+j_j=i} \psi_{j_1}\psi_{j_2}\cdots\psi_{j_j} \cdots \left\{ \begin{array}{l} =\rho_i, \text{ for } 0\le i\le n \\ \text{for } i>n, \end{array} \right. \qquad (4)$$

where $[j/n]$ stands for the integer nearest to (including j/n) but not greater than j/n. Further, ${}^{(n)}U$ can be represented by $U_0 \exp \Sigma_1^\infty \epsilon^j\, {}^{(n)}\psi_j$, where

$${}^{(n)}\psi_i = \left\{ \begin{array}{l} \sum_{j=1}^{i} \\ \sum_{j=[i/n]}^{i} \end{array} \right\} \frac{(-1)^{j+1}}{j} \sum_{j_1+\cdots+j_j=i} \rho_{j_1}\rho_{j_2}\cdots\rho_{j_j} \cdots \left\{ \begin{array}{l} =\psi_i, \text{ for } 1\le i\le n \\ \text{for } i>n. \end{array} \right. \qquad (5)$$

Note that the reciprocity, as shown by Eqs. (2) and (3), and Eqs. (4) and (5) represent basic relations that are valid for any type of wave, i.e., plane, spherical, or beam wave.

As an example of application of these formulas, specifically of Eq. (2), we will present an alternative way of obtaining the result found by Laussade and Yariv who took into account of the effect of all orders of multiple scattering in the optical wave propagation through a turbulent atmosphere.[3] It should be noted that the use of the scalar wave equation is almost completely justified for millimeter and shorter wavelengths[4] and that the results derived under the assumption that the wavelength be much smaller than the inner scale of turbulence are valid with negligible errors for microwaves as well.[5] Based on these observations it should be recognized that the scope of the problem discussed by Laussade and Yariv is to be extended to include the microwave and millimeter wavelengths.

Laussade and Yariv considered the case of a monochromatic plane wave in the positive x direction, incident upon the boundary at x = 0 of the turbulent atmosphere that occupies the region x > 0. Neglecting the $\epsilon^3\mu^3$ term they solved Eq. (1) by the Born method. (Note that ψ's used by these authors correspond to our ρ's.) With the Gaussian distribution for random variable $\mu(\mathbf{r})$ they showed under the assumption valid for optical wavelengths that $\langle \rho_{2m+1} \rangle = 0$ and $\langle \rho_{2m} \rangle = \langle \rho_2 \rangle^m / m!$. From this, they deduced

$$\langle U^{(1)}(L,\rho) \rangle = \langle U(L,\rho) \rangle \exp G(L) \qquad (6)$$

and

$$B_u^{(1)}(L,\tau) = B_u(L,\tau)\exp 2G(L), \qquad (7)$$

where

$$G(L) = \epsilon^2 k^2 L \int_0^\infty d\alpha \int dK \left(1 - \frac{k}{K^2 L}\sin\frac{K^2 L}{k}\right) F_n(K,\alpha)$$

and

$$B_u(L,\tau) = B_u(L,\rho_1,\rho_2) = \langle U(L,\rho_1)\,U^*(L,\rho_2)\rangle$$

with $\tau = \rho_1 - \rho_2$.

In Eqs. (6) and (7) we set $(x = L, y, z) = (L, \rho)$ and the superscript (1) refers to the first Rytov approximation.

Now we will show the derivation of Eqs. (6) and (7) by the use of Eq. (2) under the same Gaussian assumption for $\mu(\mathbf{r})$. We only need to study ρ_{2m} which may be written from Eq. (2) as

$$\rho_{2m} = \left(\Sigma_{q=1}^{m-1} + \Sigma_{q=m}^{2m}\right)\frac{1}{q!}\sum_{j_1+\cdots+j_q=2m}\psi_{j_1}\psi_{j_2}\cdots\psi_{j_q}.$$

By retaining the portion $\Sigma_{q=m}^{2m}$ which consists exclusively of ψ_1 and ψ_2 and neglecting the other portion $\Sigma_{q=1}^{m-1}$ which contains higher order terms $\psi_3, \cdots, \psi_{2m}$, we obtain

$$\langle\rho_{2m}\rangle \approx \frac{1}{m!}\langle\psi_2^m\rangle + \frac{1}{(m+1)!}\sum_{j_1+\cdots+j_{m+1}=2m}\langle\psi_{j_1}\psi_{j_2}\cdots\psi_{j_{m+1}}\rangle + \cdots + \frac{1}{(2m)!}\langle\psi_1^{2m}\rangle.$$

This can be shown to be approximated by

$$\langle\rho_{2m}\rangle \approx \frac{1}{m!}\left[\langle\psi_2\rangle^m + \binom{m}{1}\langle\psi_2\rangle^{m-1}\left(\frac{\langle\psi_1^2\rangle}{2}\right) + \cdots + \left(\frac{\langle\psi_1^2\rangle}{2}\right)^m\right]$$

$$= \frac{1}{m!}\left[\langle\psi_2\rangle + \frac{\langle\psi_1^2\rangle}{2}\right]^m = \frac{1}{m!}\langle\rho_2\rangle^m,$$

where use has been made of $\rho_2 = \psi_2 + \psi_1^2/2!$ that can be obtained from Eq. (2) with n = 2. Then it follows from $\langle U\rangle = U_0\sum_0^\infty \epsilon^{2m}\langle\rho_{2m}\rangle = U_0\exp\epsilon^2\langle\rho_2\rangle$ that $\langle U\rangle = [U_0\exp(\epsilon^2\psi_1^2/2)]\exp\epsilon^2\langle\psi_2\rangle = \langle U^{(1)}\rangle\exp\epsilon^2\langle\psi_2\rangle$ or $\langle U^{(1)}\rangle = \langle U\rangle\exp(-\epsilon^2\langle\psi_2\rangle)$.

From the Born and Rytov solutions to Eq. (1) we find for optical and millimeter waves that

$$\rho_1(\mathbf{r}) = \psi_1(\mathbf{r}) \approx ik\int_0^x dx'\int_K dN(K,x')\,e^{iK\cdot\rho}\,e^{-\frac{i}{2k}K^2(x-x')},$$

where we set $\mu(\mathbf{r}) = \int_K dN(K,x)\,e^{iK\cdot\rho}$.

From this we obtain

$$\frac{\psi_1^2(\mathbf{r})}{2!} \approx -\frac{k^2}{2}\int_0^x dx'\int_0^x dx''\int_K\int_{K'} dN(K,x')\,dN(K-K',x'')\,e^{iK\cdot\rho}\,e^{-\frac{i}{2k}[K'^2(x-x')+(K-K')^2(x-x'')]}.$$

In addition we find that

$$\rho_2(\mathbf{r}) \approx -k^2\int_0^x dx'\int_0^{x'} dx''\int_K\int_{K'} dN(K,x')\,dN(K-K',x'')\,e^{iK\cdot\rho}\,e^{-\frac{i}{2k}[K'^2(x-x')+(K-K')^2(x-x'')]},$$

$$\psi_2(\mathbf{r}) \approx +\frac{k^2}{2}\int_0^x dx'\int_0^x dx''\int_K\int_{K'} dN(K,x')\,dN(K-K',x'')\,e^{iK\cdot\rho}$$

$$\times\, e^{-\frac{i}{2k}[K'^2(x-x')+(K-K')^2(x-x'')]}\left(1 - e^{-\frac{i}{2k}2K'\cdot(K-K')(x-\bar{x})}\right),$$

where $\bar{x} = \max(x', x'')$. Note that $\rho_2 = \psi_2 + \psi_1^2/2!$ holds exactly. More important to note is that $\psi_2 \equiv 0$ and hence $\rho_2 = \psi_1^2/2!$ in the optical approximation where the exponential functions are set equal to unity. When this approximation is not enforced strictly, we see that ψ_2 (and hence all ψ_k, $k \geq 3$) is very small compared with ρ_2 or $\psi_1^2/2!$. This fact was used implicitly in the foregoing in evaluating $\langle\rho_{2m}\rangle$.

Moreover, it can be shown that

$$\mathrm{Re}\langle\psi_2(L,\rho)\rangle \approx -k^2 L\int_0^\infty d\alpha\int dK\left(1 - \frac{k}{K^2 L}\sin\frac{K^2 L}{k}\right)F_n(K,\alpha)$$

$$\mathrm{Im}\langle\psi_2(L,\rho)\rangle \approx -2k^3\int_0^\infty d\alpha\int\frac{dK}{K^2}\sin^2\frac{K^2 L}{2k}F_n(K,\alpha).$$

From these relations and $\langle U^{(1)}\rangle = \langle U\rangle \times \exp(-\epsilon^2\langle\psi_2\rangle)$ we can obtain Eq. (6) under suitable conditions.

In a similar manner Eq. (7) can also be derived.

References

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