Absolute Arrays of Arrays Principle

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Abstract

Schelkunoff's Arrays of Arrays Principle (AAP) is examined by using techniques from the algebraic geometry. It is shown that liner antenna arrays are naturally regarded as a function over algebraic curves and AAP is obtained from a tower of Galois coverings. Furthermore the case of infinite arrays is treated by the theory of étale fundamental groups and it ultimately gives 'Absolute Arrays of Arrays Principle.'

1. Introduction

Schelkunoff provided a view of linear antenna arrays as polynomials and formulated arrays of arrays principle where subarrays are regarded as radiating elements. Recently, the authors re-examined corresponding property by algebraic techniques and proved 'The ULA Factorization Theorem'[2]. However, the theorem only applies to Uniform Linear Arrays (ULA).

In this study, general methodology has been developed by using more sophisticated mathematics. The linear array should be regarded as a function over algebraic curves, and an array pattern factorization is naturally obtained from a tower of Galois coverings. The tower has one to one correspondence with a projective system of subgroups of additive integers. The longest sequence is obtained by a composition series of a cyclic group of order the number of radiating elements. Any subarray with arbitrary excitation can be regarded as a unique 'tree' as that in the previous work. The trees determine topology of feeding networks. This is a natural generalization of the ULA Factorization Theorem. Finally infinite arrays are examined through profinite completion procedure and 'Absolute Arrays of Array Principle' is proposed for the tower of Galois coverings.

From now on, we assume that $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ are the set of complex numbers, the real numbers, the integers, and the positive integers.

2. Scheulkunoff Polynomial and Arrays of Arrays Principle

An array factor, *i.e.* Schelkunoff polynomial $F_P(z)$, of a linear antenna array with P radiating elements is expressed as follows:

$$F_P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{P-1} z^{P-1}, \qquad (1)$$

where $a_m,\ (m = 0, 1, 2, \cdots, P-1)$ are complex array excitation coefficients. We assume that the position of m-th element is md, the observation angle is θ from the boresight, and the array is operated at the wavelength λ. Then z is given as follows:

$$z = e^{2\pi j u}, \qquad (2)$$

$$u = \frac{d}{\lambda}\sin\theta, \qquad (3)$$

where u is the universal parameter. In case that a beam is steered to an angle θ_0, the corresponding phase term can be conveniently extracted from a_m as,

$$u = \frac{d}{\lambda}(\sin\theta - \sin\theta_0). \qquad (4)$$

If $F_P(z)$ is factorized into,

$$F_P(z) = F^{(1)}_{p_1}(z) F^{(2)}_{p_2}(z^{p_1}) \cdots F^{(n)}_{p_n}(z^{p_1 p_2 \cdots p_{n-1}}), \qquad (5)$$

the AAP is readily applied by deeming the subarrays as radiating elements. We define the m-th subarray factor with K elements by the following form,

$$F^{(m)}_K(z) = a^{(m)}_0 + a^{(m)}_1 z + \cdots + a^{(m)}_{K-1} z^{K-1}, \qquad (6)$$

where $a^{(m)}_i \in \mathbb{C},\ (0 \leq i \leq K-1)$.

3. ULA Factorization Theorem

In this section, the previously reported ULA Factorization Theorem is reviewed.

Before describing it, some terminologies are defined. A finite order polynomial with the first term 1 and all the other non-zero coefficients 1 is called Unit Coefficient Polynomial (UCP), *e.g.*, $1 + x + x^5 + x^{10}$. In general, UCP corresponds to ULA with non-uniform spacings. A polynomial f in UCP is called irreducible if f cannot be factorized into products of lower order polynomials in UCP. If a polynomial f is factorized into lower order polynomial as $f = f_1 f_2 f_3 \cdots f_p$, a product of the form $f_i f_j \cdots f_q, (i, j, \cdots, q \in \{1, 2, 3, \cdots, p\})$, is called sub-product of f or shortly sub-product. With the convention, the following theorem is found,

[ULA Factorization Theorem]] Let P be a positive integer with factorization $P = p_1 p_2 \cdots p_n$, where every p_i is a prime but not necessarily distinct each other. Then the following factorization of the geometric series is possible,

$$\Psi_P(x) = \Psi_{p_1}(x)\Psi_{p_2}(x^{p_1})\Psi_{p_3}(x^{p_1 p_2}) \cdots \Psi_{p_n}(x^{p_1 p_2 \cdots p_{n-1}}). \qquad (A)$$

where,

$$\Psi_P(x) = 1 + x + x^2 + x^3 + \cdots + x^{P-1} = \frac{x^P - 1}{x - 1}.$$

The factorization has the following properties,

(i) (A) is irreducible in UCP,

(ii) Sub-product of $\Psi_P(x)$ in (A) is UCP,

(iii) If a factorization $\Psi_P(x) = \prod_i F_i$ exists, where every F_i is UCP, then every F_i is expressed by sub-product of $\Psi_P(x)$ in (A) by appropriately choosing the order of $p_1, p_2, \cdots, p_n$.

We describe an application of the theorem. We attribute a Schelkunoff polynomial $\Psi_3(x) = 1 + x + x^2$ to a tree in Fig.1. The term is an analogue of 'tree' in the discrete mathematics[3], "a loop free connected graph." If a factorization formula is given, we can create a tree in accordance with order of factorization of polynomials. For example, $\Psi_3(x)\Psi_2(x^3) = (1+x+x^2)(1+x^3)$ and $\Psi_2(x^3)\Psi_3(x) = (1+x^3)(1+x+x^2)$ correspond to Figs.2 and 3, respectively. If the order of multiplication is different, the resultant tree is different. It is noted that expanded factorization formula $\{(1+x^3)(1+x+x^2)\} = 1+x+x^2+x^3+x^4+x^5 = \Psi_6(x)$ should correspond to Fig.4. With the convention, the map from the factorization formula to the feed network is well-defined. The resultant feed network is not necessarily a tree for general array factorization formula. For example, Fig.5 corresponds to $\{\Psi_2(x)\}^2 = (1+x)^2$. The necessary and sufficient condition for the feed network being a tree is that expanded array factorization formula is UCP.

A tree is called irreducible if no finer decomposition exists. By using the map defined in the above, we have the following theorem as a corollary,

[Feed Network Factorization Theorem]] There is an isomorphism between the factorization formulas of uniformly-spaced ULA and the trees. In particular, a tree for (A) is irreducible.

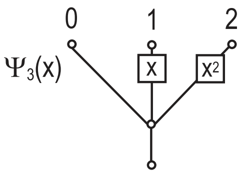

Fig. 1: $\Psi_3(x)$ tree

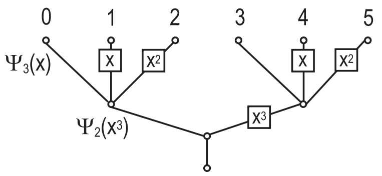

Fig. 2: $\Psi_3(x)\Psi_2(x^3)$ tree

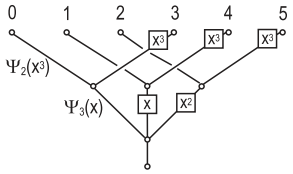

Fig. 3: $\Psi_2(x^3)\Psi_3(x)$ tree

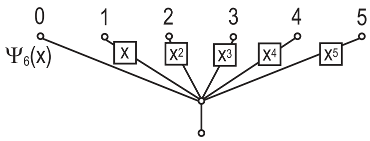

Fig. 4: $\Psi_6(x)$ tree

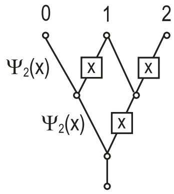

Fig. 5: $\{\Psi_2(x)\}^2$ feed network

The theorem tells us all the possible topologies of the feeding network. For example, if a uniformly-spaced ULA with 8 elements is given, allowable factorization is found to be as follows:

$$\begin{aligned}
\Psi_8(x) &= \Psi_2(x)\Psi_2(x^2)\Psi_2(x^4) = \Psi_2(x)\Psi_2(x^4)\Psi_2(x^2) \\
&= \Psi_2(x^2)\Psi_2(x)\Psi_2(x^4) = \Psi_2(x^2)\Psi_2(x^4)\Psi_2(x) \\
&= \Psi_2(x^4)\Psi_2(x)\Psi_2(x^2) = \Psi_2(x^4)\Psi_2(x^2)\Psi_2(x) \\
&= \{\Psi_2(x)\Psi_2(x^2)\}\Psi_2(x^4) = \{\Psi_2(x)\Psi_2(x^4)\}\Psi_2(x^2) \\
&= \{\Psi_2(x^2)\Psi_2(x^4)\}\Psi_2(x) = \Psi_2(x)\{\Psi_2(x^2)\Psi_2(x^4)\} \\
&= \Psi_2(x^2)\{\Psi_2(x)\Psi_2(x^4)\} = \Psi_2(x^4)\{\Psi_2(x)\Psi_2(x^2)\} \\
&= \{\Psi_2(x)\Psi_2(x^2)\Psi_2(x^4)\},
\end{aligned}$$

where product of polynomials inside the brackets $\{\cdot\}$ is regarded as being expanded.

4. Algebraic Curves and Arrays of Arrays Principle

We will try to generalize the previous theorems to the case of general linear arrays with arbitrary excitations as in (5) and (6). To achive, we need to observe that (5) is a product of functions over covering spaces of algebraic curves[4][5]:

$$\begin{aligned}
z \xrightarrow{h_{p_1}} z^{p_1} \xrightarrow{h_{p_2}} z^{p_1 p_2} \xrightarrow{h_{p_3}} \cdots \\
\cdots \xrightarrow{h_{p_{n-1}}} z^{p_1 p_2 \cdots p_{n-1}},
\end{aligned} \qquad (7)$$

where $h_n : \mathbb{C}^* \to \mathbb{C}^*;\ z \mapsto z^n$, and $\mathbb{C}^*$ is defined to be the set $\mathbb{C} - \{0\}$. Each h_n gives unbranched covering[5] and the tower corresponds to Kummer extensions[6] of algebraic functions. If we put $x = z^{p_1 p_2 \cdots p_m}$ and $y = z^{p_1 p_2 \cdots p_{m-1}} \in \mathbb{C}^*$ for the m-th part of the covering, $x - y^{p_m} = 0$ is an irreducible algebraic function. In the algebraic extension $\mathbb{C}(y,x)/\mathbb{C}(x)$, we have $\mathbb{C}(y,x) = \mathbb{C}(x)[x^{1/p_m}] \cong \mathbb{C}(x)[y]/(y^{p_m} - x) \cong \mathbb{C}(y)$, thus we can write $\mathbb{C}(y,x) = \mathbb{C}(y)$. We attribute $\mathbb{C}(x)$ the same meaning to that in the tower (7). If we write the covering space of the m-th tower $Y (\cong \mathbb{C}^*)$ and the base space $X (\cong \mathbb{C}^*)$, then $h_{p_m} : Y \to X$ is a Galois covering and corresponding fundamental gourps are $\pi_1(X) = \mathbb{Z}$ and $\pi_1(Y) = \mathbb{Z}$, respectively[5]. Indicating $\#$ as induced homomorphism by a continuous function, the Galois group $\mathrm{Gal}(Y/X)$ of the covering space is calculated with the aid of $h_{p_m \#}(\pi_1(Y)) = p_m \mathbb{Z}$ as follows:

$$\mathrm{Gal}(Y/X) = \pi_1(X)/h_{p_m \#}(\pi_1(Y)) = \mathbb{Z}/p_m\mathbb{Z}.$$

From now on, we write a m-th root of unity as $\zeta_m = \exp(2\pi j/m)$. All the roots of $x - y^{p_m} = 0$ are expressed by $y = \zeta_{p_m}^n \cdot x^{1/p_m},\ (0 \le n < p_m)$. The algebraic extension $\mathbb{C}(y)/\mathbb{C}(x)$ becomes a Galois extension in ordinary sense[6] because $\zeta_{p_m}^n : \mathbb{C}(y) \to \mathbb{C}(y),\ x^{1/p_m} \mapsto \zeta_{p_m}^n \cdot x^{1/p_m}$ gives automorphism of the field extensions where $\mathbb{C}(y) = \mathbb{C}(x)[x^{1/p_m}]$ and $\zeta_{p_m}^n \in \mathbb{C}(x)$. In this case, the Galois group $\mathrm{Gal}(\mathbb{C}(y)/\mathbb{C}(x))$ is homeomorphic to a cyclic group $\mathbb{Z}/p_m\mathbb{Z}$.

In algebraic geometry, the following equivalence is known[5],

$$\mathrm{Gal}(Y/X) \cong \mathrm{Gal}(\mathbb{C}(y,x)/\mathbb{C}(x)) \cong \mathrm{Gal}_{\mathrm{anal}}(Y/X). \tag{8}$$

$\mathrm{Gal}(Y/X)$, $\mathrm{Gal}(\mathbb{C}(y,x)/\mathbb{C}(x))$, and $\mathrm{Gal}_{\mathrm{anal}}(Y/X)$ stand for topological, algebraic, and analytical (as complex manifold) Galois groups. This is a consequence of GAGA[7]. Therefore we will freely use desired Galois groups for the rest of the manuscript.

Let us try to give physical interpretation of the action of Galois groups. If we have the following factorization,

$$F_P(z) = F_{p_1}^{(1)}(x) F_{p_2}^{(2)}(x^{lm}), \tag{9}$$

then the Galois group $\mathrm{Gal}(\mathbb{C}(x)/\mathbb{C}(x^{lm}))$ acts as $F_{p_1}^{(1)}(x) \mapsto F_{p_1}^{(1)}(x\zeta_{lm}^k)$ with the value of $F_{p_2}^{(2)}(x^{lm})$ remains unchanged. The action corresponds to change of observation 'angle' in u-space: At first trial a signal is put in the direction of the main beam and then it will be changed in the direction of a grating lobe, *i.e.* an action of the Galois group. This is a simple and natural interpretation. Even for general angles not limited to the direction of the main beam, the symmetry is valid. That is why the Schelkunoff polynomial is a function over the covering spaces.

5. Algebraic Structure of Galois Tower

In the previous section, the tower of Galois covering is considered as algebraic curves defined on $\mathbb{C}^*$'s. However the physics of the antenna arrays is adequately described on the Schelkunoff's unit circle S^1 in the complex plane. Fortunately $\mathbb{C}^*$ and S^1 are homotopy equivalent and topological properties are identical.

Let $h(y) = e^{2\pi j y} = x$ be the universal covering of $h : \mathbb{R} \to S^1$. The fiber $h^{-1}(x)$ of arbitrary point x on S^1 is isomorphic to $\mathbb{Z}$. The fundamental group $\pi_1(S^1, x)$ of the base point x acts as automorphism on $\mathbb{R}$, and is homeomorphic to $\mathbb{Z}$. Choose $y \in h^{-1}(x)$. $m \in \mathbb{Z} = \pi_1(S^1, x)$ acts as $m : y \mapsto y + m$. The action is effective on the fiber and $h : \mathbb{R} \to S^1$ is therefore a Galois covering. Owing to the Galois theory of covering spaces[5], $\mathrm{Gal}(\hat{X}/X) = \pi_1(X)$ and $X \cong \hat{X}/\mathrm{Gal}(\hat{X}/X) = \hat{X}/\pi_1(X)$ are valid for the universal covering $h : \hat{X} \to X$. In the case of $h : \mathbb{R} \to S^1$, the action of Galois group $\pi_1(S^1) = \mathbb{Z}$ to the point y results to an orbit $y + \mathbb{Z}$. The quotient space is isomorphic to $\mathbb{R}/\mathbb{Z}$ and is identified with the base space S^1. All the subgroups of $\pi_1(S^1) = \mathbb{Z}$ are of the form $m\mathbb{Z}$ with $m \in \mathbb{N}$. They determine all the possible coverings with base spaces $\mathbb{R}/(m\mathbb{Z}) \cong S^1$ by the Galois correspondence. By using the universal coverings, the corresponding tower of (7) is expressed by the following commutative diagram,

$$\begin{array}{ccccccccc}
\mathbb{R} & = & \mathbb{R} & = & \mathbb{R} & & \mathbb{R} & = & \mathbb{R} \\
{\scriptstyle h=h_1^{(u)}}\downarrow & & {\scriptstyle h_{p_1}^{(u)}}\downarrow & & {\scriptstyle h_{p_1 p_2}^{(u)}}\downarrow & \cdots & \downarrow & & {\scriptstyle h_{p_1 \cdots p_{n-1}}^{(c)}}\downarrow \\
S^1 & \xrightarrow{h_{p_1}} & S^1 & \xrightarrow{h_{p_2}} & S^1 & & S^1 & \xrightarrow{h_{p_{n-1}}} & S^1,
\end{array} \tag{10}$$

where $h_k^{(u)} : y \mapsto e^{2\pi j k y}$ and $h_k : x \mapsto x^k$, ... etc..

Let us classify the tower by algebra. First of all, note that the following proposition,

[Proposition] Let $\{p_i\}_{i \in \mathbb{N}}$ be a sequence of positive integers not necessarily distinct each other. Then arbitrary $N \in \mathbb{N}$ can be expressed by $0 \le m_i < p_i,\ (i \in \mathbb{N})$ uniquely as follows:

$$\begin{aligned} N \quad = \quad & m_1 + m_2 p_1 + m_3 p_1 p_2 + \cdots \\ & + m_n p_1 p_2 \cdots p_{n-1} + \cdots . \end{aligned} \tag{11}$$

In particular, define $N_P = N \mod P$, and then the following is valid,

$$\begin{aligned} N_{p_1 p_2 \cdots p_n} = {} & m_1 + m_2 p_1 + m_3 p_1 p_2 + \cdots \\ & + m_n p_1 p_2 \cdots p_{n-1} \mod p_1 p_2 \cdots p_n . \end{aligned} \tag{12}$$

The above representation of integers forms a projective system[8] with directed set Λ of indices $k \le l$ in case that k divides l. Its definition is as follows: Let Λ be a directed set as indices of a direct product of groups $\prod_{\lambda \in \Lambda} G_\lambda$. The $\prod_{\lambda \in \Lambda} G_\lambda$ is called a projective system if for all the $\lambda \le \mu$, there exist homomorphisms $\psi_{\lambda\mu} : G_\mu \to G_\lambda$, where $\psi_{\lambda\lambda}$ is identity, and for all $\lambda \le \mu \le \nu$, it has the properties $\psi_{\lambda\nu} = \psi_{\lambda\mu} \circ \psi_{\mu\nu}$. For example in (12), if $p_1 p_2 \cdots p_m | p_1 p_2 \cdots p_n$, then $\psi_{p_1 p_2 \cdots p_m,\ p_1 p_2 \cdots p_n} : N_{p_1 p_2 \cdots p_n} \mapsto N_{p_1 p_2 \cdots p_m}$, and we can readily observe that the the definition is satisfied. Therefore (12) forms a projective system. The tower $\mathbb{Z} \supset p_1\mathbb{Z} \supset p_1 p_2 \mathbb{Z} \supset \cdots$ of subgroups of additive integers corresponds to terms in (11). Giving a projective system is equivalent to specifying a decreasing series of the subgroups due to inclusion. The tower of S^1 in (10) has one to one correspondence with the system and all the configuration is determined by the identification.

Now, we investigate how much general the tower (7) is. The following well-known theorem[6] helps.

[Jordan=Hölder Theorem] Let G be a group having the following two composition series,

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_r = e,$$
$$G = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_s = e,$$

then $r = s$, and their quotient groups can be rearranged to be identical.

$$G_0/G_1, \quad G_1/G_2, \quad \cdots, \quad G_{r-1}/G_r,$$
$$H_0/H_1, \quad H_1/H_2, \quad \cdots, \quad H_{r-1}/H_r,$$

In particular, if we choose all the p_i's in (12) to be primes and put $P = p_1 p_2 \cdots p_n$, then the following composition series is obtained,

$$\mathbb{Z}/P\mathbb{Z} \supset p_1\mathbb{Z}/P\mathbb{Z} \supset p_1p_2\mathbb{Z}/P\mathbb{Z} \supset \cdots \supset p_1p_2\cdots p_{n-1}\mathbb{Z}/P\mathbb{Z} \supset 0. \quad (13)$$

The quotient groups are of the form $\mathbb{Z}/p_i\mathbb{Z}$. The above resolution is the longest as a composition series, so is the corresponding tower of the Galois coverings. Finally considerations given so far are summarized into a theorem,

[General Array Factorization Theorem] A Schelkunoff polynomial of a linear array with P elements can be regarded as a function over a tower of Galois coverings. Arrays of Arrays Principle is determined by a projective system of $\mathbb{Z}/P\mathbb{Z}$. In particular, the system which corresponds a composition series of $\mathbb{Z}/P\mathbb{Z}$ gives the longest tower of the Galois coverings.

The corresponding feed network theorem can be obtained by the same procedure as that in ULA case, *i.e.* by considering the sub-product of array factors as well as order of multiplications. We attribute it with the symbol $\mathcal{A}_\mathcal{P}$, and we have

$$\mathcal{A}_\mathcal{P}\left(\prod_{\lambda\in\Lambda} G_\lambda\right), \quad (14)$$

where the projective systems are those in (13).

6. Absolute Arrays of Arrays Principle

In the previous sections, finite lengths were assumed for the tower of Galois coverings, *i.e.* finite arrays. The theory will be extended to the case of infinite arrays.

The universal covering can not be constructed within the category of algebraic curves because the corresponding morphism is exponential. Grothendieck[10] discovered a method to overcome it by introducing the étale fundamental group $\pi_1^{\mathrm{et}}(X)$ which is the projective limit of a category of algebraic curves as follows:

$$\begin{aligned}\pi_1^{\mathrm{et}}(X) &= \varprojlim_{Y/X:\ \text{fin. ur. Galois}} \mathrm{Gal}(\mathbb{C}(Y)/\mathbb{C}(X)),\\ &= \mathrm{Gal}(M^{ur}/\mathbb{C}(X)),\end{aligned} \quad (15)$$

where the projective limit is taken over all the finite unramified Galois extensions of algebraic curves. $X = \mathbb{C}^*$ and $Y = \mathbb{C}^*$ in our case, and it is noted that (7) is a projective system of a tower of algebraic extensions. M^{ur} is the maximal unramified extension of the function field $\mathbb{C}(X)$, and $\mathrm{Gal}(M^{ur}/\mathbb{C}(X))$ is generally called the absolute Galois group[9]. For the universal covering $h^{(c)}$, we have $h^{(c)}: \mathbb{C} = Y \to \mathbb{C}^* = X$. It is known that the topological Galois group $\mathrm{Gal}(Y/X)$ is homeomorphic to $\pi_1(X)$, and for (15) we have the following[10],

$$\pi_1^{\mathrm{et}}(X) \cong \hat{\pi}_1(X), \quad (16)$$

where $\hat{\pi}_1(X)$ is the profinite completion by taking the projective limit of $\pi_1(X)$. In the case of the Galois coverings of linear arrays, we can define $M = \mathbb{C}(X)(S)$ with $S = \{X^{1/n} | n \in \mathbb{N}\}$ for $\mathbb{C}(X)$. Then we have $M^{ur} = M$ and the following[8][9],

$$\mathrm{Gal}(M^{ur}/\mathbb{C}(X)) = \hat{\mathbb{Z}}, \quad (17)$$

where $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, and is related to profinite completions $\mathbb{Z}_p$ (p-adic integers) with respect to primes p's as follows[8][9]:

$$\hat{\mathbb{Z}} = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z} \cong \prod_{p:\ \text{all primes in}\ \mathbb{N}} \mathbb{Z}_p. \quad (18)$$

If we choose an element of $\hat{\mathbb{Z}}$, we have the corresponding unique projective system. For finite arrays, we can regard those to be zero in the part with more than the number of radiating elements in the projective system. Now all the projective systems are contained in $\hat{\mathbb{Z}}$, and it should be regarded as the absolute Galois group of the linear antenna arrays. Accounting the trees of feeding networks with all the possible permutations including sub-product as in the previous section, we finally have the following 'Absolute Arrays of Arrays Principle,'

$$\mathcal{A}_\mathcal{P}\,\hat{\mathbb{Z}}. \quad (19)$$

7. Summary

Schelkunoff's arrays of arrays principle is naturally interpreted with a tower of Galois coverings of algebraic curves. The absolute arrays of arrays principle is formulated by the étale fundamental group with all the possible symmetries of feeding networks.

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