ELECTROMAGNETIC WAVE PROPAGATION IN BIANISOTROPIC MEDIA
a coordinate-free approach

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ABSTRACT

The object of this paper is to present a coordinate-free method in treating electromagnetic wave propagation in a bianisotropic medium. Based on the direct manipulation of vectors, dyadics and their invariants, the method eliminates the use of coordinate systems. It facilitates solutions and provides results in a greater generality. The paper contains the following results in coordinate-free forms: (1) the eigenvalue problem formulation of waves in bianisotropic media, (2) the dispersion equation, (3) the polarizations of waves in bianisotropic media, and (4) the determination of the directions of field vectors.

I. INTRODUCTION

With the rapid advance in technology, more and more materials used in applications are anisotropic. In the microwave frequency region, a ferrite is an anisotropic medium characterized by a tensor permeability and a sclar permittivity [1], while, on the other hand, a plasma is characterized by a tensor permittivity but a scalar permeability [2]. In the infrared frequency region, both permittivity and permeability of a medium can be anistoropic as occurs in Yttrium Iron Garnet (YIG) [3]. In this case we have a bianisotropic medium. In this paper, we shall present a coordinate-free method to treat wave propagation in bianisotropic media.

II. EIGENVALUE PROBLEM FORMULATION AND DISPERSION EQUATION

For a monochromatic plane wave of the form $\exp[i(\bar{k}\cdot\bar{r}-\omega t)]$, the Maxwell equations for a bianisotropic medium in a source-free region take the form

$$\omega\bar{B} = \omega\mu_o\bar{\bar{\mu}}\cdot\bar{H} = \bar{k}\times\bar{E} \qquad (1)$$

$$\omega\bar{D} = \omega\varepsilon_o\bar{\bar{\varepsilon}}\cdot\bar{E} = -\bar{k}\times\bar{H} \qquad (2)$$

where $\bar{k}$ is the wave vector, and $\bar{\bar{\mu}}$ and $\bar{\bar{\varepsilon}}$ are the relative permeability and permittivity tensors respectively. For simplicity, we assume that both $\bar{\bar{\mu}}$ and $\bar{\bar{\varepsilon}}$ are symmetric and positive definite matrices but they do not necessarily commute.

Elimination of $\bar{H}$ or $\bar{E}$ from the above equations yields

$$\bar{\bar{M}}\cdot\bar{\bar{N}}\cdot\bar{E} = \lambda\bar{E} \qquad (3)$$

or

$$\bar{\bar{N}}\cdot\bar{\bar{M}}\cdot\bar{H} = \lambda\bar{H} \qquad (4)$$

where

$$\bar{\bar{M}} = \bar{\bar{\varepsilon}}^{-1}\cdot(\hat{k}\times\bar{\bar{I}}) \qquad (5)$$

$$\bar{\bar{N}} = \bar{\bar{\mu}}^{-1} \cdot (\hat{k} \times \bar{\bar{I}}) \tag{6}$$

where $\hat{k}$ is the unit vector in the direction of wave vector $\bar{k}$. $\bar{\bar{I}}$ is a unit matrix, and the antisymmetric matrix $\hat{k} \times \bar{\bar{I}}$ is defined according to $(\hat{k} \times \bar{\bar{I}}) \cdot \bar{C} = \hat{k} \times \bar{C}$ for any vector $\bar{C}$ [4].

According to Eq. (3), $\bar{E}$ is an eigenvector of matrix $\bar{\bar{M}} \cdot \bar{\bar{N}}$, corresponding to the eigenvalue $\lambda = -1/n^2$, where $n = ck/\omega$ is the index of refraction. Similarly, Eq. (4) shows that $\bar{H}$ is an eigenvector of matrix $\bar{\bar{N}} \cdot \bar{\bar{M}}$, corresponding to the same eigenvalue λ.

Furthermore, Eqs. (1) and (2) yield the eigenvalue problems for $\bar{D}$ and $\bar{B}$:

$$(\bar{\bar{M}} \cdot \bar{\bar{N}})^T \cdot \bar{D} = \lambda \bar{D} \tag{7}$$

and

$$(\bar{\bar{N}} \cdot \bar{\bar{M}})^T \cdot \bar{B} = \lambda \bar{B} \tag{8}$$

where the superscript T denotes the transpose of the matrix. Thus $\bar{D}$ and $\bar{B}$ are the eigenvectors of the transposes of matrices $\bar{\bar{M}} \cdot \bar{\bar{N}}$ and $\bar{\bar{N}} \cdot \bar{\bar{M}}$, respectively.

In terms of the refractive index vector $\bar{n} = n\hat{k}$, Eq. (3) may be rewritten as

$$[\bar{\bar{\varepsilon}} + (\bar{n} \times \bar{\bar{I}}) \cdot \bar{\bar{\mu}}^{-1} \cdot (\bar{n} \times \bar{\bar{I}})] \cdot \bar{E} = 0 \tag{9}$$

which will have nontrivial solution $\bar{E}$ provided that the determinant of the coefficient matrix vanishes; that is

$$|\bar{\bar{\varepsilon}} + (\bar{n} \times \bar{\bar{I}}) \cdot \bar{\bar{\mu}}^{-1} \cdot (\bar{n} \times \bar{\bar{I}})| = 0 \tag{10}$$

Expanding the above determinant according to the Cayley-Hamilton Theorem [4], we obtain the dispersion equation in a coordinate-free form:

$$An^4 - Bn^2 + C = 0 \tag{11}$$

where

$$A = (\hat{k} \cdot \bar{\bar{\varepsilon}} \cdot \hat{k}) \ (\hat{k} \cdot \bar{\bar{\mu}} \cdot \hat{k})$$
$$B = \hat{k} \cdot \bar{\bar{\mu}} \cdot \{[(\mathrm{adj}\,\bar{\bar{\varepsilon}}) \cdot \bar{\bar{\mu}}]_t \bar{\bar{I}} - (\mathrm{adj}\,\bar{\bar{\varepsilon}}) \cdot \bar{\bar{\mu}}\} \cdot \hat{k}$$
$$C = |\bar{\bar{\mu}} \cdot \bar{\bar{\varepsilon}}|$$

Eq. (11) shows that the index of refraction n depends on the direction of wave propagation $\hat{k}$.

III. POLARIZATIONS OF WAVES

Now let us examine the polarizations of the field vectors in a bianisotropic medium. Taking the cross product of Eq. (2) with its complex conjugate, and using the identity

$$(\mathrm{adj}\ \bar{\bar{A}}) \cdot (\bar{u} \times \bar{v}) = (\bar{\bar{A}}^T \cdot \bar{u}) \times (\bar{\bar{A}}^T \cdot \bar{v}) \tag{12}$$

we obtain

$$\begin{aligned} \bar{D} \times \bar{D}^* &= \varepsilon_o^2 \ (\mathrm{adj}\ \bar{\bar{\varepsilon}}) \cdot (\bar{E} \times \bar{E}^*) \\ &= (1/\omega^2)\ \bar{k}\bar{k} \cdot (\bar{H} \times \bar{H}^*) \end{aligned} \tag{13}$$

similarly, from Eq. (1), we have

$$\bar{B}\times\bar{B}^* = \mu_o^2 \ (\text{adj } \bar{\bar{\mu}})\cdot(\bar{H}\times\bar{H}^*)$$
$$= (1/\omega^2)\ \bar{k}\bar{k}\cdot(\bar{E}\times\bar{E}^*) \tag{14}$$

or

$$\bar{H}\times\bar{H}^* = (1/\omega^2\mu_o^2|\bar{\bar{\mu}}|)\ \bar{\bar{\mu}}\cdot\bar{k}\bar{k}\cdot(\bar{E}\times\bar{E}^*) \tag{15}$$

Eqs. (13, (14), and (15) show that $\bar{D}\times\bar{D}^*$, $\bar{B}\times\bar{B}^*$ and $\bar{H}\times\bar{H}^*$ vanish when $\bar{E}\times\bar{E}^*$ vanishes. In other words, if the vector $\bar{E}$ is linearly polarized ($\bar{E}\times\bar{E}^* = o$), so are the field vectors $\bar{D}$, $\bar{B}$, and $\bar{H}$.

Substituting Eq. (15) into Eq. (13), we find that the vector $\bar{E}\times\bar{E}^*$ satisifies the following homogeneous equation:

$$\{\text{adj } \bar{\bar{\epsilon}} - [(\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n})/|\bar{\bar{\mu}}|]\bar{n}\bar{n}\}\cdot(\bar{E}\times\bar{E}^*) = o \tag{16}$$

Thus, for nonzero vector $\bar{E}\times\bar{E}^*$ to exist, the determinant of the coefficient matrix of Eq. (16) must vanish; that is,

$$(\bar{n}\cdot\bar{\bar{\epsilon}}\cdot\bar{n})(\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n}) - |\bar{\bar{\mu}}\cdot\bar{\bar{\epsilon}}| = o \tag{17}$$

Hence we can conclude that the plane waves in a bianisotropic medium are always linearly polarized except when the refractive index vector $\bar{n}$ satisifies condition (17). In that case, the wave can have any polarization. Condition (17) implies that the discriminant $\Delta = B^2-4AC$ of the dispersion Eq. (11) is equal to zero. That is, when the refractive index vector $\bar{n}$ satisifies the condition (17), the two solutions in n^2 of the dispersion equation become equal. Those directions of the wave normal that cause the discriminant Δ to vanish and yield two equal roots for n^2 are called the optic axes of the bianisotropic media. In summary, aside from the optic axes, for each direction of wave normal $\hat{k}$, there are (in general) two linearly polarized waves propagating at different phase velocities in a bianisotropic medium.

IV. DIRECTIONS OF FIELD VECTORS

For a given refractive index vector $\bar{n} = n\hat{k}$, the field vectors of the plane wave in a bianisotropic medium are completely determined. To show this, we dot-premultiply Eq. (9) by $\text{adj}\bar{\bar{\mu}}$ and obtain

$$[(\text{adj}\bar{\bar{\mu}})\cdot\bar{\bar{\epsilon}} - (\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n})\bar{\bar{I}}]\cdot\bar{E} = -(\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{E})\bar{n} \tag{18}$$

Thus the direction of $\bar{E}$ depends on whether the matrix $[(\text{adj}\bar{\bar{\mu}})\cdot\bar{\bar{\epsilon}} - (\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n})\bar{\bar{I}}]$ is singular or nonsingular. In the nonsingular case, we dot-premultiply Eq. (18) by the inverse of $[(\text{adj}\bar{\bar{\mu}})\cdot\bar{\bar{\epsilon}} - (\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n})\bar{\bar{I}}]$, and obtain the direction of $\bar{E}$ as

$$\bar{e} = \{\text{adj } [(\text{adj}\bar{\bar{\mu}})\cdot\bar{\bar{\epsilon}} - (\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n})\bar{\bar{I}}]\}\cdot\bar{n} \tag{19}$$

The directions of other field vectors follow from the Constitutine relations and the Maxwell equations:

$$\bar{d} = \epsilon_o\bar{\bar{\epsilon}}\cdot\{\text{adj}[(\text{adj}\bar{\bar{\mu}})\cdot\bar{\bar{\epsilon}} - (\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n})\bar{\bar{I}}]\}\cdot\bar{n} \tag{20}$$

$$\bar{b} = (1/c)\ (\bar{n}\times\bar{e}) \tag{21}$$

$$\bar{h} = (1/\mu_o)\bar{\bar{\mu}}^{-1}\cdot\bar{b} \tag{22}$$

On the other hand, if the matrix $[(\mathrm{adj}\bar{\bar{\mu}})\cdot\bar{\bar{\epsilon}}-(\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n})\bar{\bar{I}}]$ is singular, or $\lambda = (\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n})/|\bar{\bar{\mu}}|$ is an eigenvalue of the matrix $\bar{\bar{A}} = \bar{\bar{\mu}}^{-1}\cdot\bar{\bar{\epsilon}}$, the inverse of the coefficient matrix does not exist. In this case, we may express the dispersion equation and the adjoint matrix of $(\bar{\bar{A}}-\lambda\bar{\bar{I}})$ as

$$\hat{k}\cdot\bar{\bar{\mu}}\cdot\mathrm{adj}[(\mathrm{adj}\bar{\bar{\mu}})\cdot\bar{\bar{\epsilon}}-(\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n})\bar{\bar{I}}]\cdot\hat{k} = o \tag{23}$$

and

$$\mathrm{adj}(\bar{\bar{A}}-\lambda\bar{\bar{I}}) = C\bar{u}\bar{u}\cdot\bar{\bar{\epsilon}} \tag{24}$$

respectively, where C is an arbitrary constant and $\bar{u}$ is an eigenvector of $\bar{\bar{A}}$ corresponding to the eigenvalue λ.

Substitution of Eq. (24) into Eq. (23) yields the condition imposed on the eigenvector $\bar{u}$:

$$\bar{u}\cdot\bar{\bar{\epsilon}}\cdot\hat{k} = o \tag{25}$$

From Eq. (4), we obtain

$$[|\bar{\bar{\mu}}\cdot\bar{\bar{\epsilon}}|-(\bar{n}\cdot\bar{\bar{\epsilon}}\cdot\bar{n})(\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n})](\bar{u}\cdot\bar{B}) = o \tag{26}$$

To satisify condition (26), two possibilities arise: $\bar{u}\cdot\bar{B} = o$ or $\bar{u}\cdot\bar{B} \neq o$. In the first case ($\bar{u}\cdot\bar{B} = o$), $\bar{B}$ is perpendicular to $\bar{u}$. But, according to Maxwell's equations, $\bar{B}$ must also be perpendicular to $\bar{n}$. Thus the direction of $\bar{B}$ is parallel to their cross product:

$$\bar{b} = \bar{u}\times\bar{n} \tag{27}$$

The directions of the remaining vectors are determined by the Maxwell equations and constitutive relations:

$$\begin{aligned}\bar{h} &= (1/\mu_o)\bar{\bar{\mu}}^{-1}\cdot(\bar{u}\times\bar{n})\\ \bar{d} &= -(1/c)(\bar{n}\times\bar{h})\\ \bar{e} &= -(1/\epsilon_o c)\bar{\bar{\epsilon}}^{-1}\cdot(\bar{n}\times\bar{h})\end{aligned} \tag{28}$$

In the second case ($\bar{u}\cdot\bar{B} \neq o$), we must have

$$|\bar{\bar{\mu}}\cdot\bar{\bar{\epsilon}}|-(\bar{n}\cdot\bar{\bar{\epsilon}}\cdot\bar{n})(\bar{n}\cdot\bar{\bar{\mu}}\cdot\bar{n}) = o \tag{29}$$

This is the condition (17) denoting the coincidence of the given wave normal with the optic axes. In this case, the field vectors are limited only by the conditions implied by the Maxwell equations.

V. REFERENCES

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