ASYMPTOTIC AND NUMERICAL TECHNIQUES FOR EVALUATING FOURIER AND HANKEL TRANSFORMS IN ELECTROMAGNETICS

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ABSTRACT

This paper discusses the evaluation of integral transforms with a Fourier or a Fourier-Bessel (Hankel) kernel. It is shown that, for integrals defined on the real axis, integration by parts and consideration of some canonical integrals provides directly asymptotic expansions (AE) for these transforms. These techniques replace advantageously the more involved steepest descent method in the complex plane.
Several techniques are discussed for the numerical evaluation of these integrals. A new algorithm, exploiting optimally the asymptotic behaviour of the integrands is introduced. Extensive tests show that the new technique is far more accurate that currently used algorithms like the Clenshaw-Curtiss rules or the Romberg adaptative quadrature combined with Shanks extrapolation.

INTRODUCTION

Integral transforms offer now a widespread approach for solving applied electromagnetic problems. Among the most useful stand the Fourier and Hankel transforms, which share the property of an oscillatory kernel and are included in the general expression

$$I(q) = \int_a^b f(x)\, g(qx)\, dx \tag{1}$$

where :

- [a,b] is a segment of the real axis x
- g(qx) is an oscillating function [exp(-jqx), $H_0^{(2)}(qx)$ and/or their real and imaginary parts]
- f(x) is a well-behaved non-oscillating function, which may increase with qx (but not exponentially). For the sake of simplicity f(x) is assumed to be real. Complex functions can be handled by investigating, sequentially, their real and imaginary parts.

In most practical situations, the upper limit of integration is infinite, $b=\infty$. This prevents the use of standard numerical integration techniques and calls for specially tailored methods. In the following sections, asymptotical expansions (AE) for I(q) will be constructed and a new numerical technique based upon these AE will be described.

ASYMPTOTIC TECHNIQUES

Integration by parts has seldom been used in the litterature. However, AE for the integrals represented in (1) can easily be obtained with this technique [1].

When applied to expression (1) in the particular case $g(qx) = \exp(jqx)$, integration by parts gives the following AE :

$$I(q) = \frac{1}{jq} \exp(jqx) \sum_{0}^{\infty} (j/q)^n f^{\underline{|n}}(x) \Big|_a^b \qquad (2)$$

where $f^{\underline{|n}}(x)$ stands for $d^n f/dx^n$ and $f^{\underline{|0}}(x) = f(x)$.
Taking the real and imaginary parts in (2), AE for the cases $g(qx) = \cos qx$, $\sin qx$ are easily obtained. When the upper limit is infinite, it must be recalled that $\lim(b\to\infty) \exp(jqb) = 0$ in the sense of distributions. Hence, for the interval $[a,\infty]$, we obtain :

$$\int_a^{\infty} f(x) \exp(jqx)\, dx = \frac{j}{q} \exp(jqa) \sum_{0}^{\infty} (\frac{j}{q})^n f^{\,n}(a) \qquad (3)$$

In deriving expressions (2) and (3), implicit use has been made of the continuity of $f(x)$ and of its derivatives troughout the integration interval. If $f(x)$ exhibits some kind of singularity, the expressions (2) and (3) are no longer valid. The easiest way to overcome this drawback is to consider a canonical integral having an analytical solution and including the same type of singularity.
For instance, to treat infinite derivatives at some point $x=p$ (which correspond to branch points in the complex plane) the following integrals are useful

$$I_1 = \int_0^p (p^2-x^2)^{\frac{1}{2}} \exp(jqx)\, dx \quad , \quad I_2 = \int_p^{\infty} (x-(x^2-p^2)^{\frac{1}{2}}) \exp(jqx)\, dx$$

The exact solutions for I_1 and I_2 are listed in 2 . Their AE are :

$$I_1 \to jp/q + \pi p\, H_1^{(1)}(pq) \,/\, 2q \quad , \quad I_2 \to -p \exp(jqp) \,/\, jq + \frac{j\pi p}{2q} H_1^{(1)}(pq)$$

Comparing these AE with the values predicted by equation (2) it is concluded that the terms involving Hankel functions give the specific contribution of the infinite derivative at $x=p$.
Once the case $g(qx)$ $\exp(jqx)$ has been completely studied, the extension to other kernels like Bessel or Hankel functions is easily made, replacing them by their Fourier transforms.

NUMERICAL TECHNIQUES : THE WEIGHTED MEANS ALGORITHM

Asymptotical techniques only provide accurate results for large values of the parameter q . For smaller q, numerical techniques must be used.
Let us introduce the notation :

$$I_n^{(0)} = \int_{a_0}^{a_n} f(x) \exp(jqx)\, dx \quad , \quad a_n = a_0 + n\pi/q \quad , \quad a_0 = a \qquad (4)$$

The sequence $I_n^{(0)}$ approaches the true value $I = I_\infty^{(0)}$ when n increases, and truncation error can be estimated, by using (3) as :

$$I - I_n^{(0)} = \int_{a_n}^{\infty} = \frac{j}{q} \exp(jqa_n) \,(f_n + \frac{j}{q} f'_n + ..) \qquad (5)$$

where f_n and f'_n are shorthand notations for $f(x=a_n)$ and $df/dx|_{x=a_n}$

Since $\exp(jqa_{n+1}) = -\exp(jqa_n)$, a better estimation of I is given by the weighted mean:

$$I_{n+\frac{1}{2}}^{(1)} = \frac{I_n^{(0)} + A\, I_{n+1}^{(0)}}{1 + A} = I + \left[\frac{f'_n - A\, f'_{n+1}}{1 + A}\right] \frac{\exp(jqx_n)}{q^2} \quad , \; A = \frac{f_n}{f_{n+1}} \qquad (5)$$

In practice, if f(x) shows an asymptotic behaviour of the type Cx^{α}, the f_n values can be replaced by n^{α} and, therefore, we get :

$$I_{n+\frac{1}{2}}^{(1)} = (I_n^{(0)} + A\, I_{n+1}^{(0)})/(1 + A) \quad , \quad A = (n/(n+1))^{\alpha} \qquad (6)$$

Under the same assumtions, a careful evaluation of the bracketted term in (5) shows that it is proportionnal to $(n+\frac{1}{2})^{\alpha-2}$. Hence, a new sequence defined by

$$I_n^{(2)} = (I_{n-\frac{1}{2}}^{(1)} + A\, I_{n+\frac{1}{2}}^{(1)})/(1 + A) \quad , \quad A = ((n-\tfrac{1}{2})/(n+\tfrac{1}{2}))^{\alpha-2} \qquad (7)$$

will yield a better approximation to I . By applying recursively the formulas (6),(7),..., reducing every time the exponent α by a factor two, it is possible to extract from an original sequence one unique value which is the best estimation for I that the sequence can provide.

RESULTS : THE VAN DER POL'S INTEGRAND

The above techniques have been applied to a Hankel transform arising in the study of wave propagation above the earth. This transform has an analytical solution due to Van der Pol [3] and can be written :

$$(8) \qquad I = \int_0^{\infty} \frac{2}{u_1+u_2} J_0(\lambda\rho)\, \lambda\, d\lambda = \frac{2}{\rho} \frac{\partial}{\partial\rho}[(\exp(-jk_2\rho) - \exp(-jk_1\rho))/\rho]/(k_1^2-k_2^2)$$

with $u_i^2 = k_i^2 - \lambda^2$.

The real parts of the integrand $J_0(\lambda\rho)f(\lambda)$, and of the "envelop" $f(\lambda)$ have been plotted in fig.1 . Infinite derivatives at $\lambda= k_1,k_2$ are readily observable. The above described asymptotic techniques give the AE :

$$I \rightarrow \frac{2j}{\rho^2} [k_2 \exp(-jk_2\rho) - k_1 \exp(-jk_1\rho)]/(k_2^2-k_1^2) \qquad (9)$$

whose correctness can be checked by direct inspection of (8).
Fig.2 gives the relative errors obtained when evaluating (8) by numerical techniques. For the same number of evaluations, the weighted means give far more accurate results that the commonly used Romberg-Shanks algorithm [4]. Moreover, improvement with the number of integration points is more sensible with the weighted means. Even if accuracy deteriorates when the parameter ρ increases, precision remains better than 0.01 % in the $0<\rho<10$ interval and for only 56 evaluations of the integrand.

REFERENCES [1] : N.BLEINSTEIN, R.A. HANDELSMAN."Asymptotic evaluation of integrals",Holt,Rinehart and Winston, New York, 1975
[2] : I.GRADSTEYN, I.RYZHIK."Tables of Series, Integrals and Products" Academic Press, New York, 1965.
[3] : A. BANOS."Dipole radiation in the presence of a conducting half-space" Pergamon Press, Oxford, 1966.
[4] : R.J. LYTLE, D.L. LAGER."Numerical evaluation of Sommerfeld integrals" Report UCRL-52423, Lawrence Livermore Laboratory, University of California, 1974.

Figure 1 : Real part of Van der Pol's integrand. (Zero value for $\lambda<k_1$).
$\rho = 2,\ k_1 = 1,\ k_2 = 2$.

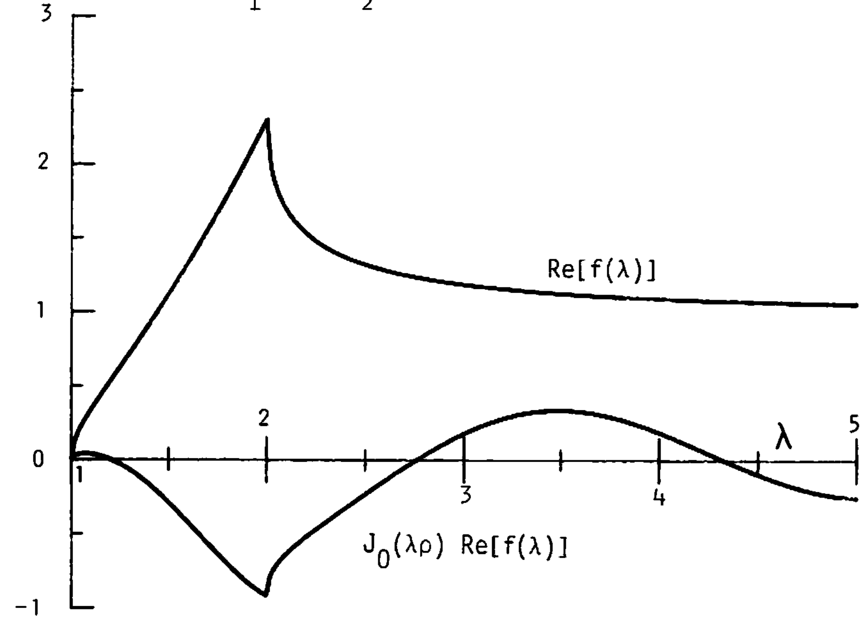

Figure 2: Relative error E in the numerical evaluation of Van der Pol's integral as a function of parameter ρ .
WM 40, WM 56 : Weighted means with 40,56 integration points
R-S 40, R-S 56 : Romberg-Shanks with 40,56 integration points