THEORY OF SCATTERING BY AN OPEN BOUNDARY

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ABSTRACT. A general and rigorous mathematical theory to analyse diffraction and scattering of waves by an open boundary is given. Basing on the theory, a numerical method is derived which is useful and effective to solve various practical problems.

INTRODUCTION. A two-dimensional "open boundary" is a set of open segments of straight or curved lines of finite length in a plane. (Fig.1). Similarly, a three-dimensional one is a set of open segments of surfaces of finite dimensions which may have slots in them. (Fig.2) (For a detailed definition, the reader is referred to [3].)

Such an open boundary is a model of a set of thin metallic walls, and diffraction and scattering of waves by such walls is analysed by a boundary value problem for the Helmholtz equation or the Maxwell equations for the corresponding open boundary. Hence, a study of problems for an open boundary is extremely important in Electromagnetic theory, and in Antenna theory as well.

Fig.1

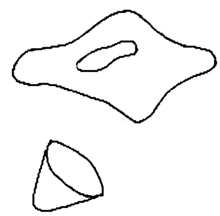

Fig.2

Though many works have been done on boundary value problems in Electromagnetic theory, boundaries dealt with in most of them were closed one, and to the best knowledge of the author, there had existed no rigorous and general theory on a problem for an open boundary before the author's works [1,2,3,4].

A closed boundary, such as a circle or a sphere, is the one which separates the whole space into two parts, an interior and an exterior. As soon as a part of a closed boundary is removed, the interior domain disappears immediately while sharp edges appear in the boundary, and it becomes an open boundary. This simple fact of disappearance of an interior domain and appearance of edges makes it very difficult to solve a boundary value problem. For example, a problem with respect to a circle is easily solved, however, it becomes very difficult if the circle has apertures in it. We shall study further to know why problems for an open boundary are so difficult.

Most general and complete works for a closed boundary may be the ones by H.Weyl [5], A.P.Calderón [6], and K.Yasuura [7]. Weyl reduced the problem for a closed boundary to that of solving for an integral equation of Fredholm of the second kind, which was then studied in the realm of the theory of L_2-space. However, as was proved by the author [3], the reduction to the equation of the second kind was possible because an interior domain existed, and the L_2-theory was applicable since no edges existed in the closed boundary. That is, Weyl's method is not applicable if a boundary is open. Furthermore, as is shown by Theorem 1 below, the problem for an open boundary is equivalent to that of solving for an integral equation of the first kind, of which a few has been known and whose solution is not necessarily in L_2. In other words, the useful results known on an equation of the second kind and on the L_2-space are of no use in our case, and hence, we have to construct a new theory of an equation of the first kind, considering that a solution may be a Schwartz's distribution. This is the first subject of this paper.

Calderón and Yasuura, independently to each other, established a "mode expansion method." Yasuura studied the two-dimensional Helmholtz equation for a closed boundary C, and showed that $v_n = \Sigma c_k u_k$, $(k=1,2,\text{--},n)$, where u_k are pertinent solutions of the Helmholtz equation and c_k are constants, tend to a true solution v of the boundary value problem uniformly in a domain outside of C, provided v_n tend to a given boundary value g on C. He employed as solutions u_k the ones obtained by the method of separation of variables, which, in the case of an exterior problem, satisfy the radiation condition but are not bounded at the origin located in the interior of C, and which, in the case of an interior problem, are bounded in the interior of C but not satisfy the radiation condition. Yasuura's method was supported by the facts that (i) the space $\{u_k\}$ is dense in the space $\{g\}$ on C, and (ii) a solution v depends on the boundary data g continuously, which have been proved with the help of the integral equation of the second kind concerning the boundary C.

Since Yasuura's "mode functions" u_k are either unbounded at the origin or non-radiating at infinity, they are not adoptable in our case where no interior exists. Moreover, (i) and (ii) above have not been proved in this case. Therefore, we have to construct new mode functions u_k so that they meet the following rather severe requirements; (iii) u_k satisfy the Helmholtz equation, (iv) u_k are bounded everywhere and satisfy the radiation condition simultaneously, (v) the space $\{u_k\}$ is dense in the space of boundary data $\{g\}$ on an open boundary. Also, we must prove that (vi) a solution v of the boundary value problem for an open boundary depends on the boundary value g continuously. It is the second purpose of this paper to show that we can realize (iii)$\sim$(vi) above, and then, basing on the results thus obtained, to derive a numerical method which is useful to solve various practical problems.

Our fundamental equation is (2); $\Psi\tau = g$, and most of difficulties mentioned above come from the fact that $\Psi\tau \to 0$ does not necessarily mean $\tau \to 0$. (Riemann-Lebesgue's theorem.) This fact also implies that a direct numerical calculation of (2) is not permitted, since a small change in g, say a truncation error, may cause a big variation in a solution τ.

As a conclusion of all what was mentioned above, we see that the full of the mathematical theory described in the following section is indispensable in both of a theoretical and a numerical analysis of the boundary value problem for an open boundary.

THEORY. Owing to the limitted pages of paper, only a theory of the Dirichlet problem for the two-dimensional Helmholtz equation is summarized. Note that the similar holds also for the three-dimensional Helmholtz equation and for the Maxwell equations as well.

Denote an open boundary on a plane $\mathbb{R}^2$ by L, points on $\mathbb{R}^2$ by x,y, etc., the distance between x and y by $|x-y|$, end points of L by x^*, and set $\psi(x,y) = (1/4j)H_0^{(2)}(k|x-y|)$, where $H_0^{(2)}$ is the zero-th order Hankel function of the second kind and k is a wave constant. Let

u be the component, perpendicular to $\mathbf{R}^2$, of an electric field diffracted by L, then it (a) satisfies the Helmholtz equation, (b) assumes a given boundary value g on L, (c) satisfies the radiation condition at ∞, and (d) satisfies the edge condition at x^*. Then,

Theorem 1. A function u satisfying (a)∿(d) above is necessarily represented as

$$(1) \qquad u(x)=\int_L \psi(x,y)\tau(y)ds_y - u_0(x), \qquad x\notin L.$$

where u_0 is given in terms of g, while τ should satisfy the following integral equation of Fredholm of the first kind;

$$(2) \qquad \Psi\tau \equiv \int_L \psi(x,y)\tau(y)ds_y = g(x), \qquad x\in L.$$

Conversely, if a solution τ of (2) is found and if u is defined by (1) in terms of τ, then, it satisfies all of (a)∿(d) above. That is, the analysis of diffraction of electric field by L is equivalent to that of solving for the equation (2).

Theorem 2. $\Psi\tau=0 \;\rightleftarrows\; \tau=0.$

Theorems 1 and 2 have been proved in [3]. On differentiating (2), the author converted it to a singular integral equation, and then solved it. For details, the reader is referred to [3].

As usual, let C be the space of continuous functions on L, C^∞ be the space of infinitely many times differentiable functions on L, and C_0^∞ be the subspace of C^∞ whose elements are zero in a vicinity of end points x^*. Since $\psi(x,y)$ has a log.singularity at $x=y$, $\Psi\tau$, which is a function of $x\in L$, is once (tangentially) differentiable but not differentiable twice or more even if $\tau\in C^\infty$. That is, the smoothness of τ is not inherited to $\Psi\tau$. However, as is proved by Theorem 3 below, the smoothness inherits if $\tau\in C_0^\infty$ and we have $\Psi\tau\in C^\infty$.

Set $\psi(x,y)=\psi^{[o]}(x,y)=\psi_{(o)}(x,y)=O(\log|x-y|)$, and define $\psi_{(m)}$ and $\psi^{[m]}$ successively by $\psi_{(m)}(x,y)=\int^y \psi_{(m-1)}(x,z)ds_z$, where the integral is an indefinite (line) integral along L till $y\in L$. and $\psi^{[m]}(x,y)=(d/ds_x)\int^y \psi^{[m-1]}(x,z)ds_z$, where d/ds_x is the tangential differentiation along L with respect to $x\in L$. Then, we can prove

$$(3) \quad \psi^{[m]}(x,y)=\{(-1)^m/2\pi\}\psi(x,y)+O(1), \quad (d/ds_x)^m\psi_{(m)}(x,y)=\psi^{[m]}(x,y).$$

Set $\hat{\sigma}=\Psi\sigma$ where $\sigma\in C_0^\infty$, and let $\sigma^{(m)}=\sigma^{(m)}(x)$ and $\hat{\sigma}^{(m)}=\hat{\sigma}^{(m)}(x)$ be the m-th (tangential) derivatives of σ and $\hat{\sigma}$ with respect to x, respectively. On integrating by parts, we have

Theorem 3. $\hat{\sigma}^{(m)}=\hat{\sigma}^{(m)}(x)=(-1)^m\int_L \psi^{[m]}(x,y)\sigma^{(m)}(y)ds_y.$

Since $\sigma^{(m)}\in C$, (3) and Theorem 3 shows that $\hat{\sigma}^{(m)}\in C$ for $m=0,1,2,\cdots$. That is, $\hat{\sigma}\in C^\infty$, and Ψ maps $\sigma\in C_0^\infty$ into $\hat{\sigma}\in C^\infty$. Consequently, if we set $\hat{\Sigma}\equiv\{\hat{\sigma}\,|\,\hat{\sigma}=\Psi\sigma,\sigma\in C_0^\infty\}$, we have $\hat{\Sigma}\subset C^\infty$. The following is one of the most important theorems in our research, but it's proof is too long to describe here.

Theorem 4. $\sigma\to 0$ in $C_0^\infty \;\rightleftarrows\; \hat{\sigma}=\Psi\sigma\to 0$ in C^∞.

Here, "$\sigma\to 0$ in C_0^∞" means that σ and it's derivatives of all order tend to zero uniformly on L. "$\hat{\sigma}\to 0$ in C^∞" is defined similarly. With help of Theorem 4, togetherwith Hahn-Banach's theorem and Riesz's theorem in the theory of function spaces, we can show that

Theorem 5. $\hat{\Sigma}$ is a closed subset of C^∞, and is dense in C^∞.

Therefore, we have $\hat{\Sigma}=C^\infty$. This result, togetherwith Theorem 2, implies that, for any $\hat{\sigma}\varepsilon C^\infty$, there exists a unique $\sigma\varepsilon C_0^\infty$ such that $\hat{\sigma}=\Psi\sigma$. In other words, the equation (2) has the unique solution $\tau\varepsilon C_0^\infty$ if $g\varepsilon C^\infty$. However, this is not the case for a wider class of g. For example, if g merely belongs to C, then, τ must be understood to be a Schwartz's distribution.

Assume D is a bounded, closed domain in $\mathbf{R}^2$ which is outside of L. Suppose $x'\varepsilon D$ and $x\varepsilon L$. Since $x'\neq x$, $\psi(x',x)\varepsilon C^\infty$ with respect to x, and hence, by Theorem 5, there exists a function $\gamma(x,x')\varepsilon C_0^\infty$ such that $\psi(x',x)=\int_L\psi(x,y)\gamma(y,x')ds_y$. From this expression, it is proved that γ is the Green function concerning an open boundary L, and that the solution v of the Dirichlet problem for L and for a boundary value g is represented as $v(x')=\int_L\gamma(y,x')g(y)ds_y$, which proves the existence of the solution v in the same time.

Let L_2 be the space of functions which are square integrable on L, and set $(f,g)=\int_L f(x)\overline{g(x)}ds_x$ and $\|g\|^2=(g,g)$ for $f,g\varepsilon L_2$.

Then, from the last result, we have

(4) $$\sup|v(x')|\leqq\|g\|\cdot\sup\{\int_L|\gamma(y,x')|^2ds_y\}^{1/2}$$

which proves the continuous dependence of v on g.

Assume $\{\phi_k\}$ is a complete orthonormal system of functions in the space L_2, and set $u_k=\Psi\phi_k$. Note that $u_k(x')$, and hence $v_n(x')=\Sigma c_k u_k(x')$, where c_k are constant, $(k=1,2,--,n)$, satisfy the Helmholtz equation as functions of $x'\varepsilon D$. Furthermore, when $x'\to x\varepsilon L$, $v_n(x')$ tends to $g_n(x)\equiv v_n(x)=\Sigma c_k\Psi\phi_k(x)$ which is continuous on L.

Let a function $g\varepsilon C$ and positive constants δ,δ' be given arbitrarily. It is known that there exists a function $\hat{\sigma}\varepsilon C^\infty$ such that $\|\hat{\sigma}-g\|<\delta$ holds. By Theorem 5, there exists $\sigma\varepsilon C_0^\infty$ such that $\hat{\sigma}=\Psi\sigma$. σ is expanded in a Fourier series as $\sigma=\Sigma c_k\phi_k$, where $c_k=(\sigma,\phi_k)$, of which $\|\sigma-\sigma_n\|<\delta'$ holds for a pertinent n and $\sigma_n=\Sigma c_k\phi_k$, $(k=1,2,---,n)$.

As a consequence of what mentioned above, we have a conclusion; For an arbitrarily given boundary value $g\varepsilon C$, there correspond $\hat{\sigma}$, σ and it's Fourier coefficients $\{c_k\}$. If we set $v_n(x')=\Sigma c_k u_k(x')$, $(k=1,2,---,n)$, $v_n(x')$ satisfies the Helmholtz equation at $x'\varepsilon D$ and assumes the boundary value $g_n(x)\equiv v_n(x)=\Psi(\Sigma c_k\phi_k)=\Psi\sigma_n$, of which we have $\|g-g_n\|\leqq\|g-\hat{\sigma}\|+\|\Psi(\sigma-\sigma_n)\|<\delta+\|\Psi\|\delta'$. Therefore, by (4), we have $\sup|v(x')-v_n(x')|\leqq\|g-g_n\|\cdot\sup\{\int_L|\gamma(y,x')|^2ds_y\}^{1/2}\to 0,\ (n\to 0)$.

As was proved above, v_n tends to the true solution v. However, if n is prescribed and if the best approximation is looked for, constants c_k are determined so that $\|g-g_n\|$ assumes the minimum value. That is, they are determined so as to satisfy the following simultaneous linear equations

(5) $$\sum_{k=1}^{n}(u_k,u_m)c_k=(g,u_m),\qquad m=1,2,---,n.$$

(5) was applied to the case of a plane wave incident to a straight line segment of a finite length, and obtained a numerical result which showed a very good coincidence with the exact solution obtained in terms of the Mathieu functions.

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