Proceedings of ISAP'04, Sendai, JAPAN

# SOR Algorithm for Solving Matrix Equation in MoM Analysis of Periodic Structures

Qiang Chen, Qiaowei Yuan‡, and Kunio Sawaya
Tohoku University, Sendai, Japan
‡Intelligent Cosmos Research Institute, Sendai, Japan

## 1 Introduction

It is required to develop an efficient analysis method to investigate the characteristics of the large-scale periodic structures such as the large-scale array antennas and metameterials having a periodic structure of small resonant particles and interesting properties such as the negative permittivity and the negative magnetic permeability.

The method of moment (MoM) is one of the efficient methods for the EM analysis of the periodic structure. When the array antenna or the periodic structure is composed of $N$ elements and each element is divided into $M$ segments for sub domain MoM analysis, $N_T \times N_T$ matrix equation has to be solved to obtain the unknown current vector, where $N_T = M \times N$. When the direct method such as the Gauss-Jordan method is employed to solve the matrix equation, the CPU time is proportional to $N_T^3$. In the case of a large-scale periodic structure, $N$ becomes so large that the CPU time for solving the matrix equation is much longer than that for evaluating the impedance matrix, which is proportional to $N_T^2$, and becomes the dominant part of the total CPU time in the MoM analysis [1].

Instead of direct methods, iterative methods such as the Gauss-Seidel method and the Conjugate Gradient (CG) method have been applied to solve the linear matrix equations. The number of arithmetic operations of these iterative methods is usually proportional to $N_T^2$ for each iteration step. However, it was pointed out that the required number of the iteration step of the CG method depends on the analysis model and the size of segments of the basis functions, and is usually proportional to $N_T$, which means that the total number of arithmetic operation is proportional to $N_T^3$, the same order to the direct method [2, 3]. On the other hand, the criterion of the Gauss-Seidel method is too strict to be applied directly in the MoM analysis [4], [5], but this problem has been solved by using the sub matrix as the iteration unit in stead of the matrix element [5].

In this paper, an iterative algorithm based on the successive overrelaxation (SOR) technique is introduced to solve the matrix equation in the MoM analysis of the periodic structures, whose CPU time is approximately proportional to $N_T^2$ and is even faster than the Gauss-Seidel method. The convergence criterion of the iterative algorithm is investigated and the effectiveness of the method is shown by numerical examples.

## 2 Iterative Method

The impedance matrix $[Z]$ in the MoM is divided into $[S]$ and $[T]$ so that the matrix equation of $[Z][I] = [V]$ becomes

$$[S][I] = -[T][I] + [V], \tag{1}$$

where $[S]$ consists of the lower-left triangular part including the diagonal elements of $[Z]$ , and $[T]$ consists of the upper-right triangular part excluding the diagonal elements. In the present iterative algorithm, the periodic structure is divided into a number of groups and each group consists of several neighboring periodic elements, so that the impedance matrix can be decomposed into a number of sub matrices corresponding to the group of the elements. The diagonal sub matrices $[\overline{Z}]_{ii}$ in the impedance matrix describe the self and mutual impedance between the divided unknown segments in the group $i$, and the off-diagonal sub matrices $[\overline{Z}]_{ij}$

ISBN: 4-88552-206-4 C3055©IEICE

include the mutual impedance between two divided unknown segments of different groups $i$ and $j$. The sub matrices are the basic iteration units rather than the matrix element in the ordinary SOR iteration method. When the total array elements are divided into $N/K$ groups completely, where $N$ is the total number of elements and $K$ is the number of elements in each group, the iterating procedure is expressed by:

$$\left[\bar{I}^t\right]_i^{(l+1)} = \left[\bar{I}\right]_i^{(0)} - \left[\overline{Z}\right]_{ii}^{-1} \left[\sum_{j=1}^{i-1} \left[\overline{Z}\right]_{ij} \left[\bar{I}\right]_j^{(l+1)} + \sum_{j=i+1}^{N/K} \left[\overline{Z}\right]_{ij} \left[\bar{I}\right]_j^{(l)}\right]^H, \tag{2}$$

and

$$[\bar{I}]_i^{(l+1)} = [\bar{I}]_i^{(l)} + \omega \left( \left[\bar{I}^t\right]_i^{(l+1)} - [\bar{I}]_i^{(l)} \right), i = 1, 2, \cdots, N/K; l = 0, 1, \cdots, L. \tag{3}$$

where $[\bar{I}]_i$ is a $MK$ current vector of the group $i$ and $[\bar{I}^t]_i$ is a $MK$ vector for storing $[\bar{I}]_i$ temporarily. $[\bar{I}]_i^{(0)}$ is the initial value of the current vector of group $i$ evaluated neglecting the mutual coupling between the groups and calculated by

$$\left[\bar{I}\right]_i^{(0)} = \left[V^{inc}\right]_i / \left[\overline{Z}\right]_{ii}, \tag{4}$$

where $[V^{inc}]_i$ is the incident voltage vector of group $i$. $\omega$ is the overrelaxation parameter, which should be properly determined to accelerate the convergence. The SOR method becomes the Gauss-Seidel method when $\omega$=1. The iteration continues until the criterion

$$|I_k^{(L)} - I_k^{(L-1)}| / |I_k^{(L-1)}| \leq \epsilon \tag{5}$$

is satisfied for a small value of $\epsilon$, and for all the dipole segments ($k = 1, 2, \cdots, MN$) at the final $L$th step, where $I_k$ is the current on the dipole segment $K$.

## 3 Numerical Results

The piece-wise sinusoidal (PWS) MoM [6] is applied to investigate the convergence characteristics of the iteration method. Fig. 1 shows analysis model which is an $N$-element 1-D array anntena of dipole elements with length of $2h$, radius of $a$, and array spacing of $d$. Each dipole element is divided into $M$ overlapping dipole segments for the subdomain MoM analysis. In the following numerical analysis, $h$=0.25$\lambda$, $a$=2.5×10$^{-3}\lambda$, and $M$=9.


Figure 1: Analysis model: a linear dipole array antenna.

The optimized value of $\omega$ is investigated to show the required iteration steps $L$ when $\omega$ varies and $\epsilon$ in the convergence criterion shown in Equation (5) is $1 \times 10^{-8}$. Fig. 2 shows the number of required iteration steps when array spacing $d$ is changed. It is found that the iteration steps for the case of $\omega$=0.8 is less than that for $\omega$=0.2 when $d$ is larger than about 0.5$\lambda$. However, the number of the required iteration steps increases so rapidly when $d$ decreases that more iteration steps are required for $\omega$=0.8 when $d$ is smaller than 0.3$\lambda$. It indictes that a large $\omega$ can accelerate the iteration when the mutual coupling between the neighboring groups is small, but could make

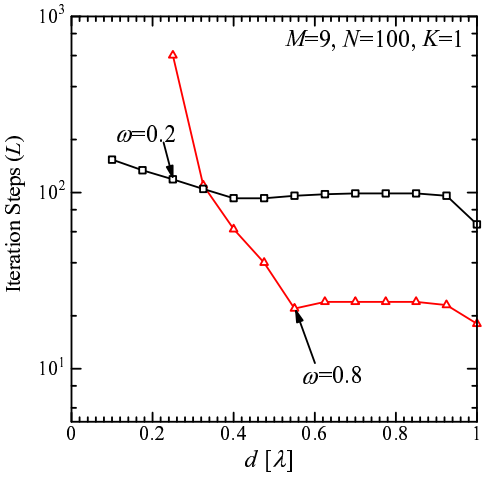


Figure 2: Required iteration steps versus array spacing.

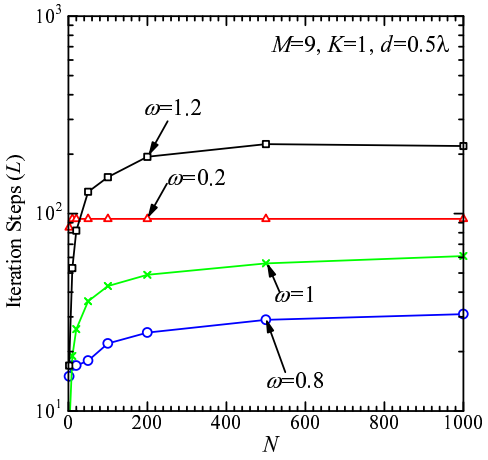


Figure 3: Required iteration steps versus number of array elements.

the iteration unstable and degrade the convergence when the mutual coupling is strong or the array has a large number of elements. Since it has been shown that the grouping technique can reduce the mutual coupling between the groups and make the convergence stable [5], selecting a relatively large $\omega$ together with the grouping technique could accelerate the convergence.

The number of iteration steps versus the total number of the dipole array elements is shown in Fig. 3. The number of iteration steps becomes almost independent of the element number $N$ when $N$ is large. The total CPU time $T$ for solving the matrix of MoM can be estimated by the expression $T = \alpha(KM)^3 + \beta L(MN/K)^2$ where the first term is for evaluating $[\bar{Z}]_{ii}^{-1}$ and the second term is for each iteration step. $\alpha$ and $\beta$ are constants depending on the computer performance. If the number of iteration steps $L$ is independent of the number of the elements $N$, the computational cost would be approximately proportional to $N^2$ when $N$ is as large that the value of second term is much greater than the value of the first term.

The residual norm of the evaluated current distribution at the final iteration step $L$ is evaluated for various $\omega$ and shown in Fig. 4, where the value of $L$ is the same as shown in Fig. 2. The residual norm is smaller than $2 \times 10^{-7}$ for all cases, which indicates that good accuracy of the iteration is obtained.

In order to demonstrate that the proposed method is effective to save the CPU time for solving the matrix equation in MoM with a large number of unkowns, the CPU time versus $N$ is shown in Fig. 5. In the calculation, the total unknown is from 90 to 9000 since each element is divided into 9 dipole segments. The value of the CPU time was measured on PC with CPU of Pentium-IV 2GHz. The curve of the Gauss-Jordan method is also plotted in the same figure for comparison. In Fig. 5, $\omega$=0.8 is the most effective parameter to save the CPU time when $N$ is in the range of 20 to 1000, while $\omega$=1.2 is the most time consuming value compared with the other values of $\omega$. The results show that the CPU time is proportional to $N^3$ by using the Gauss-Jordan method, while it is almost proportional to $N^2$ by using the present method. It is also found that the CPU time in the case of $\omega$=0.8 is about half of the Gauss-Seidel method ($\omega$=1). The cost saving effect of the iteration method is significant for a problem with large number of unknowns.

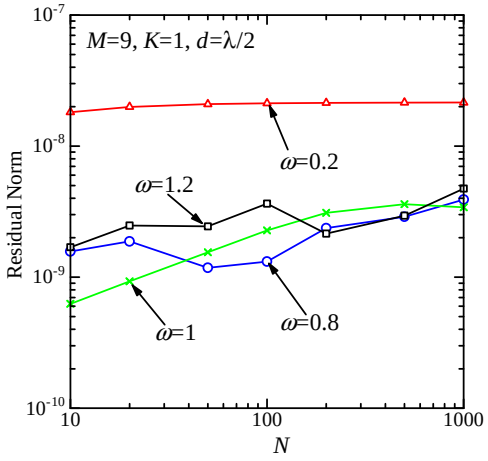


Figure 4: Residual norm of evaluated current versus array element number.

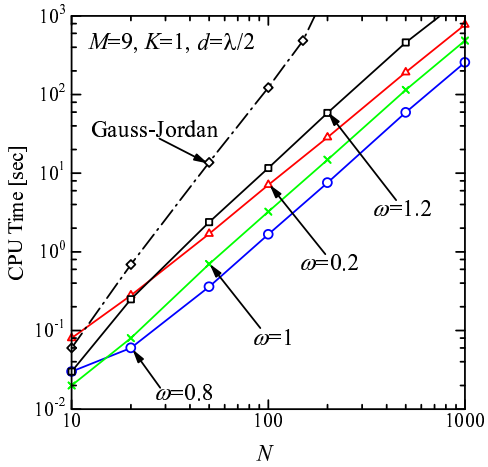


Figure 5: CPU time versus array element number.

## 4 Conclusions

A SOR iterative algorithm has been proposed to solve the matrix equation of the MoM analysis for the periodic structures. The convergence criterion of the iterative algorithm has been investigated numerically. The CPU time has been shown to be approximately proportional to $N^2$ when $N$ is large enough, which is greatly reduced compared with a direct method such as the Gauss-Jordan method. The method is faster the Gauss-Seidel by selecting a proper $\omega$. The optimum value of $\omega$ depends on the geometry of the array elements, array spacing, scan angle and so on.

## References

[1] Tapan K. Sarkar, Kenneth R. Siarkiewicz, and Roy F Straton, "Survey of Numerical Methods for solution of Large Systems of Linear Equations for Electromagnetic Field Problems," *IEEE Trans. Antennas Propagat.*, vol. AP-29, pp. 847-856, Nov. 1981.

[2] Andrew F. Peterson and Raj Mittra, "Convergence of the Conjugate Gradient Method when Applied to Matrix Equations Representing Electromagnetic Scattering Problems," *IEEE Trans. Antennas Propagat.*, Vol. AP-34, pp. 1447-1454, Dec. 1986.

[3] A. F. Peterson, C. F. Smith and R. Mittra, "Eigenvalues of Moment-Method Matrix and Their Effect on Conjugate Gradient Algorithm," *IEEE Trans. Antennas Propagat.*, Vol. AP-36, pp. 1177-1179, Aug. 1988.

[4] Matthew N. O. Sadiku, *Numerical Techniques in Electromagnetics*, Second Edition, CRC Press, 2001.

[5] Q. Chen, Q. Yuan, K. Sawaya, "Fast Algorithm for Solving Matrix Equation in MoM Analysis of Large-scale Array Antennas", *IEICE Trans. Commun.* vol. E85-B, No.11, pp.2482-2488, Nov. 2002.

[6] Jack. H. Richmond and N. H. Greay, "Mutual impedance of non-planar-skew sinusoidal dipoles, " *IEEE Trans. Antennas and Propagat.*, vol.AP-23, pp.412-414, May 1975.