NUMERICAL ACCURACY OF IDTD METHOD

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Abstract

An integro-difference time-domain (IDTD) method, which is based on the integral form of Maxwell's equations, has been recently developed to achieve 4^{th} order accuracy in space and time by taking into account the spatial and temporal variations of electromagnetic fields within each computational cell [1]. The IDTD method maintains the same numerical structure and Courant-Friedrichs-Lewy (CFL) stability criteria as the conventional second-order FDTD method. In this paper, we present the unique features of IDTD and some numerical results that demonstrate the higher order accuracy and superior dispersion properties of the IDTD method.

I. Introduction

The initial FDTD algorithm developed by Yee [2] was obtained by using the differential form of Maxwell's curl equations and is second order accurate both in space and in time. It is well recognized, however, that Maxwell's integral equations can be useful in analyzing spatial variations of electromagnetic fields within each computational cell, and some algorithms have been developed in an attempt to characterize these spatial variations more accurately [3]. Most of these algorithms are second order accurate and become identical to Yee's original FDTD difference equations in the computational domain away from a material interface.

Recently, an integro-difference time-domain (IDTD) method was introduced that exhibits fourth order (4, 4) accuracy in space and time. The IDTD method uses the integral form of Maxwell's equations and its higher order accuracy is obtained by taking into account the spatial and temporal variations of electromagnetic fields within each computational cell. In the algorithm, the electromagnetic fields within each cell are represented by space and time integrals (or integral averages) of the fields, i.e., the electric and magnetic fluxes $(\boldsymbol{D}, \boldsymbol{B})$ are represented by the surface-integral average, and the electric and magnetic fields $(\boldsymbol{E}, \boldsymbol{H})$ by the line and time integral average. The integral average fields in the staggered update equations are then associated with constitutive relations for these fields.

The numerical structure of the IDTD method remains the same as the conventional second-order update equations and, more importantly, does not require the storage of field variables at the previous time steps to obtain the fourth order accuracy in time. It is these integral average fields that are updated by the IDTD difference equations, thus numerical integration is not necessary to perform integral averages.

Dispersion analysis indicates that the IDTD algorithm exhibits superior dispersion properties compared to the conventional FDTD method, while the Courant-Friedrichs-Lewy (CFL) stability criterion of IDTD is shown to be the same as the conventional FDTD method.

In this paper we present these unique features of the IDTD algorithm and present some numerical examples that demonstrate the numerical accuracy and superior dispersion properties of the IDTD method. It will be shown that the IDTD method is significantly more accurate than other fourth order (2, 4) accurate FDTD methods [4]-[5], not to mention the standard second-order FDTD methods.

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II. The Integro-Difference Time-Domain (IDTD) METHOD

The IDTD update equations are obtained by considering the integral form of Maxwell's equations over a computational cell on the Yee computational lattice [2].

Specifically, Maxwell's integral equations yield the following space-time integro-difference equations [1],

$$\begin{aligned}
\left\langle\left\langle D_z(z_k,t_{n+1/2})\right\rangle\right\rangle_{i,j} &= \left\langle\left\langle D_z(z_k,t_{n-1/2})\right\rangle\right\rangle_{i,j} \\
&-\frac{\Delta t}{\Delta y}\left[\left\langle\left\langle H_x(y_{j+1/2},z_k)\right\rangle\right\rangle_i^n - \left\langle\left\langle H_x(y_{j-1/2},z_k)\right\rangle\right\rangle_i^n\right] \\
&+\frac{\Delta t}{\Delta x}\left[\left\langle\left\langle H_y(x_{i+1/2},z_k)\right\rangle\right\rangle_j^n - \left\langle\left\langle H_y(x_{i-1/2},z_k)\right\rangle\right\rangle_j^n\right] - \Delta t\left\langle\left\langle\left\langle J_z(z_k)\right\rangle\right\rangle\right\rangle_{ij}^n . \\
\left\langle\left\langle B_x(x_i,t_{n+1})\right\rangle\right\rangle_{j-1/2,k} &= \left\langle\left\langle B_x(x_i,t_n)\right\rangle\right\rangle_{j-1/2,k} \\
&+\frac{\Delta t}{\Delta z}\left[\left\langle\left\langle E_y(x_i,z_{k+1/2})\right\rangle\right\rangle_{j-1/2}^{n+1/2} - \left\langle\left\langle E_y(x_i,z_{k-1/2})\right\rangle\right\rangle_{j-1/2}^{n+1/2}\right] \\
&-\frac{\Delta t}{\Delta y}\left[\left\langle\left\langle E_z(x_i,y_j)\right\rangle\right\rangle_k^{n+1/2} - \left\langle\left\langle E_z(x_i,y_{j-1})\right\rangle\right\rangle_k^{n+1/2}\right].
\end{aligned} \tag{1}$$

Here $\left\langle A(y,z,t)\right\rangle_i$ is the line integral average of a scalar field $A(x,y,z,t)$, $\left\langle\left\langle A(x,t)\right\rangle\right\rangle_{i,j}$ is the surface-integral average, and $\left\langle A(\boldsymbol{r})\right\rangle^n$ is the time integral average. Note that the IDTD update equations are given in terms of line, surface and time integral averages over space-time coordinates in a computational lattice.

The IDTD method uses the leapfrog scheme where the surface-integral averaged electric displacements are updated in time. After each update, it will be necessary to change the surface-integral averaged fields to the line and time integral averaged fields in order to update the magnetic flux. The constitutive relations that relate the surface-integral averaged fluxes (*D, B*) to the line and time integral averaged fields (*E, H*) are given by [1]

$$\varepsilon\left\langle\left\langle E_z(x_i,y_j)\right\rangle\right\rangle_k^{n+1/2} \cong \left\langle\left\langle D_z(z_k,t_{n+1/2})\right\rangle\right\rangle_{i,j} + \frac{1}{24}\left(\delta_n^2+\delta_k^2-\delta_i^2-\delta_j^2\right)\left\langle\left\langle D_z(z_k,t_{n+1/2})\right\rangle\right\rangle_{i,j}. \tag{2}$$

$$\mu\left\langle\left\langle H_x(y_{j+1/2},z_k)\right\rangle\right\rangle_i^n \cong \left\langle\left\langle B_x(x_i,t_n)\right\rangle\right\rangle_{j+1/2,k} + \frac{1}{24}\left(\delta_n^2+\delta_i^2-\delta_j^2-\delta_k^2\right)\left\langle\left\langle B_x(x_i,t_n)\right\rangle\right\rangle_{j+1/2,k}, \tag{3}$$

The second terms in the right side of (2) and (3) are the higher order correction due to spatial and temporal variations of D_z within a unit cell in space-time coordinates. We note that the temporal variation $\delta_n^2 D_z$ in (2) can be obtained in terms of the spatial variations $\delta_i^2 D_z, \delta_j^2 D_z,$ and $\delta_k^2 D_z$ via the wave equation. For example, we can represent $\delta_n^2 D_z$ as

$$\delta_n^2 D_z(x_i,y_j,z_k,t_{n+1/2}) = (\Delta t)^2\left(\frac{1}{\varepsilon\mu}\right)\left(\frac{\delta_i^2 D_z}{(\Delta x)^2}+\frac{\delta_j^2 D_z}{(\Delta y)^2}+\frac{\delta_k^2 D_z}{(\Delta z)^2}\right) \tag{4}$$

This representation of δ_n^2 in terms of spatial variations reduces computational resources significantly since it is not necessary to store field variables in memory at three or more successive time steps to obtain δ_n^2 by finite-difference approximations.

III. IDTD Dispersion Relation and Stability Condition

We consider electromagnetic waves that are governed by the IDTD update equations and constitutive relations - equations (1), (2), and (3). Following a standard dispersion analysis, we can obtain the dispersion relation for IDTD as [1],

$$\frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{(c\Delta t)^2} = A_{yy}A_{zz}\frac{\sin^2\left(\frac{k_x\Delta x}{2}\right)}{(\Delta x)^2} + A_{zz}A_{xx}\frac{\sin^2\left(\frac{k_y\Delta y}{2}\right)}{(\Delta y)^2} + A_{xx}A_{yy}\frac{\sin^2\left(\frac{k_z\Delta z}{2}\right)}{(\Delta z)^2}. \qquad (5)$$

where

$$A_{xx} = 1-\frac{1}{6}\left[\left(1+\left(\frac{c\Delta t}{\Delta x}\right)^2\right)\sin^2\left(\frac{k_x\Delta x}{2}\right)-\left(1-\left(\frac{c\Delta t}{\Delta y}\right)^2\right)\sin^2\left(\frac{k_y\Delta y}{2}\right)-\left(1-\left(\frac{c\Delta t}{\Delta z}\right)^2\right)\sin^2\left(\frac{k_z\Delta z}{2}\right)\right],$$

$$A_{yy} = 1-\frac{1}{6}\left[\left(1+\left(\frac{c\Delta t}{\Delta y}\right)^2\right)\sin^2\left(\frac{k_y\Delta y}{2}\right)-\left(1-\left(\frac{c\Delta t}{\Delta z}\right)^2\right)\sin^2\left(\frac{k_z\Delta z}{2}\right)-\left(1-\left(\frac{c\Delta t}{\Delta x}\right)^2\right)\sin^2\left(\frac{k_x\Delta x}{2}\right)\right], \qquad (6)$$

$$A_{zz} = 1-\frac{1}{6}\left[\left(1+\left(\frac{c\Delta t}{\Delta z}\right)^2\right)\sin^2\left(\frac{k_z\Delta z}{2}\right)-\left(1-\left(\frac{c\Delta t}{\Delta x}\right)^2\right)\sin^2\left(\frac{k_x\Delta x}{2}\right)-\left(1-\left(\frac{c\Delta t}{\Delta y}\right)^2\right)\sin^2\left(\frac{k_y\Delta y}{2}\right)\right].$$

The brackets in (6) contain the higher order correction terms due to the spatial and temporal variations of electromagnetic fields within a computational cell.

It is well known that the CFL stability criterion can be obtained directly from the dispersion relation. After a straightforward analysis, the CFL condition for the IDTD algorithm is obtained as [1],

$$c\Delta t_{IDTD(4,4)} \le \left[\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}+\frac{1}{(\Delta z)^2}\right]^{-1/2}. \qquad (7)$$

This stability criteria is identical to the stability limits of Yee's second order FDTD scheme based on the differential formulation [2], and it is more lenient than the stability limit of any other higher order FDTD schemes such as Fang's or Lan's higher order FDTD schemes $\Delta t = (6/7)\Delta t_{FDTD(2,2)}$ [4]-[5].

IV. Numerical Results

We simulate the propagation of electromagnetic waves excited by a differentiated Gaussian point source. The current source is located at the center of the three dimensional computational domain and is given by

$$J_z = \frac{d}{dt}\exp\left(-\left(\frac{t-t_0}{t_w}\right)^2\right)^{-1/2}. \qquad (8)$$

Here t_0 is the time at the center of the Gaussian pulse and t_w is the Gaussian width. We have chosen $t_0 = 40\Delta t$ and $t_w = 5\Delta t$ where the time step Δt is chosen to be 0.04 nsec. In the simulation, we have used a seven-layer conventional PML boundary condition located three cells away from the field point.

The E_z field component is obtained at the field point $(0, 0.25\,\mathrm{m}, 0)$ on the *y*-axis as a function of time. In our results, we have used the CFL criterion (7) for the (4,4) IDTD and second order FDTD, and the stricter and narrower time step, $\Delta t = (6/7)\Delta t_{IDTD}$, for the fourth order (2,4) FDTD.

In order to compare and analyze the numerical accuracy between different algorithms, we computed the field error in the computed E_z field, which is defined as the absolute value of the difference between the analytical and numerical results, with the grid size $\Delta x = \Delta y = \Delta z = 0.025\,\text{m}$. The field errors corresponding to the E_z field is shown in Fig. 1. It clearly indicates that the field errors are significantly reduced by the higher order algorithms, and the numerical error of (4,4) IDTD algorithm is approximately 60 % to 70 % of that of the (2,4) FDTD method. We note that the IDTD numerical accuracy is achieved even with a larger time stencil by a factor of 7/6 of the (2,4) FDTD results. The superior numerical accuracy of (4,4) IDTD algorithm is due to the fact that the IDTD algorithm is also fourth order accurate in time as opposed to the (2,4) FDTD.

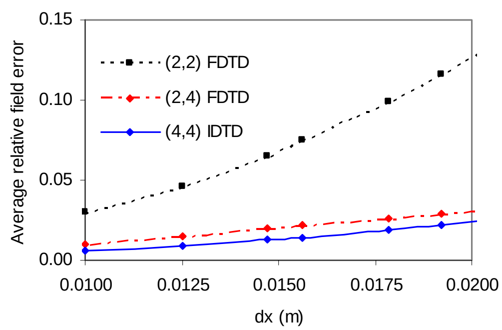

Fig. 1. Average relative field errors: second-order FDTD (dotted) and fourth-order FDTD (dashed) and the IDTD (solid line).

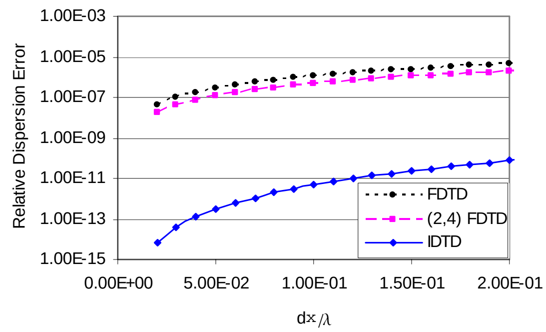

Fig. 2. Relative dispersion error: second-order FDTD (dotted) and fourth-order (dashed) FDTD methods, IDTD (solid line).

Fig. 2 shows the dispersion error of the waves propagating along the grid axis. The result indicates that the dispersion error for the IDTD case is negligible compared with the second order and fourth order (2,4) FDTD dispersion errors.

These numerical results demonstrate that the fourth-order IDTD method exhibits superior numerical accuracy to other fourth order (2, 4) FDTD methods. Furthermore the IDTD method is shown to be numerically efficient since it maintains the same numerical structure and uses the same CFL stability criterion as the conventional second order FDTD method.

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