C–2–3 MATHEMATICAL PROPERTIES AND PHYSICAL INTERPRETATION OF THE SOURCE DYADIC OF ELECTRIC DYADIC GREEN'S FUNCTIONS

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The electric dyadic Green's function, unlike the magnetic dyadic Green's function and the Green's functions of linear circuit theory, requires the specification of two dyadics, the conventional dyadic outside the source point and a source dyadic $\bar{\bar{L}}$ which is determined solely from the geometry of the "principal volume" which is chosen to exclude the source point [1-4]. Specifically, the electric field $\bar{E}(\bar{r})$ can be written [3]

$$\bar{E}(\bar{r}) = i\omega\mu_o \lim_{V_\varepsilon \to 0} \int_{V-V_\varepsilon} \bar{\bar{G}}_e(\bar{r},\bar{r}')\cdot\bar{J}(\bar{r}')dV' + \frac{\bar{\bar{L}}\cdot\bar{J}}{i\omega\varepsilon_o} , \qquad (1)$$

where V_ε is the principal volume which excludes the singularity of the conventional electric dyadic Green's function $\bar{\bar{G}}_e(\bar{r},\bar{r}')$ when $\bar{r}$ is in the source region. The source dyadic $\bar{\bar{L}}$ is defined by an integration over the surface S_ε (with unit normal $\hat{n}$) of the principal volume:

$$\bar{\bar{L}} \equiv \frac{1}{4\pi} \int_{S_\varepsilon} \frac{\hat{n}\bar{R}}{R^3}\, dS , \qquad (\bar{R} = \bar{r}'-\bar{r}) . \qquad (2)$$

The contribution from the dyadic $\bar{\bar{L}}$, which has been evaluated and tabulated for a number of principal volumes [3], is of practical importance since it is essential to the correct handling of the singularity in the numerical analysis of many problems [5,6].

The first property of $\bar{\bar{L}}$ derived from eq. (2) is that the value of $\bar{\bar{L}}$ depends only on the shape and orientation of the principal volume surface S_ε and the position of the singularity ($\bar{R}$=0) within the principal volume. Secondly, eq. (2) is expanded and manipulated in rectangular coordinates to prove that $\bar{\bar{L}}$ is always a symmetric dyadic regardless of how unsymmetric the principal volume. And thirdly, the trace of the dyadic $\bar{\bar{L}}$ is shown to equal unity for an arbitrary principal volume. These three mathematical properties serve as helpful checkpoints in verifying the analytical or numerical evaluation of $\bar{\bar{L}}$ from eq. (2) for any principal volume.

From a mathematical point of view, $\bar{\bar{L}}$ defined by eq. (2) has the form of a generalized or dyadic solid angle normalized by $1/4\pi$. That is, if a dot product were placed between $\hat{n}$ and $\bar{R}$ in eq. (2), the integration would be that of a solid angle integrated over the closed surface S_ε, resulting in a value of 4π steradians, regardless of the geometry of the principal volume.

With no dot between $\hat{n}$ and $\bar{R}$, the integration can still be considered to have units of steradians, but the result is the symmetric, unit-trace dyadic $\bar{\bar{L}}$ whose value does depend on the geometry of the principal volume. An appropriate mathematical name for $\bar{\bar{L}}$ might be the "normalized dyadic solid angle."

The dyadic $\bar{\bar{L}}$ has an elegant physical interpretation which lends important insight into the mathematical results. Suppose one were to measure the electric field at a point within a current distribution by removing an infinitesimally small volume of current and inserting an ideal point probe. The measured field would be that given by eq. (1) but without the $\bar{\bar{L}}$ term, since the current $\bar{J}$ at this point has been removed. This measured or "local" field would also depend upon the shape of the infinitesimal volume and its relative position and orientation with respect to the point probe, because it can be shown that each term separately in eq. (2) (and thus the first term) displays said dependence. Thus $\bar{\bar{L}}$ determines the perturbation in electric field caused by the hypothetical measurement scheme of removing an infinitesimally small volume of current V_ε of given shape, and position and orientation with respect to an ideal point probe, which measures the electric field.

The perturbation of electric field caused by the removal of an infinitesimal volume of dielectric material is a familiar phenomenon in electrostatics. And, in fact, if the electrostatic field is correctly expressed in terms of polarization within dielectric material, a source dyadic identical to $\bar{\bar{L}}$ appears. Thus, eqs. (1) and (2) can be used to generalize the electrostatic concept of local field to time-harmonic fields.

Finally, the actual size, in practice, of the principal volume V_ε required to compute accurately the integral in eq. (1) is discussed.

References

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