C–2–2

ON THE CONCEPT OF "REACTION" FOR ELECTROMAGNETIC FIELDS IN DISPERSIVE MEDIA

Masahiro Agu

Department of Electronics, Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki, Japan.

SYNOPSIS

The concept of "reaction" in vacuum electromagnetic theory is developed systematically to electromagnetic fields in general dispersive media. The reaction is introduced as a measure of the amount of the interactions between field source and field detector. It is found that the equations of the reaction field are adjoint to those of the electromagnetic fields. The Lagrangian formulations of the electromagnetic fields are obtained by treating the reaction fields as the conjugate fields to the electromagnetic fields. Here the Lagrangians are interpreted as the variational expressions of the mount of the interactions of field source with field detector. A reciprocity theorem having clear-cut physical meaning is derived with the use of the concept of "reaction field".

1. Introduction

The concept of "reaction" in vacuum electromagnetic theory has been introduced as a physical observable representing the amount of interactions between a field source and some other field source. The concept has been used to simplify the boundary value problems or to obtain variational expressions in the scattering problems.

The aim of the present article is to extend systematically the concept of "reaction" to the case of the electromagnetic fields in general dispersive media.

2. "Reaction fields" in Dispersive Media

Let us consider harmonic electromagnetic fields in dispersive media. The Maxwell equations for the harmonic fields with anguler frequency ω are assumed to be

$$\nabla \times \mathbb{E}(\mathbb{r}) = -j\omega\mu_0 \mathbb{H}(\mathbb{r}), \tag{1}$$

$$\nabla \times \mathbb{H}(\mathbb{r}) = j\omega(\varepsilon_{\alpha\beta})\mathbb{E}(\mathbb{r}) + \mathbb{J}(\mathbb{r}), \tag{2}$$

where $\varepsilon_{\alpha\beta}(\omega)$ is a permittivity tensor of temporally dispersive medium and, for simplicity, its spatial dispersion is not taken into account. The electric field $\mathbb{E}(\mathbb{r})$ at point $\mathbb{r}$ which is produced by the current source $\mathbb{J}(\mathbb{r}')$ located at $\mathbb{r}'$ is represented with the use of the dyadic green function,

$$E_\alpha(\mathbb{r}) = \int d\mathbb{r}' G_{\alpha\beta}(\mathbb{r}\,|\,\mathbb{r}') J_\beta(\mathbb{r}'), \tag{3}$$

where the suffixes α, β mean x, y, z and the repeated indices are assumed to be summed. Taking into account the fact that the "reaction" has been introduced into the vacuum electromagnetic theory as a measure of the interactions between a field source and some other field source,[1] we define the "reaction" $E^+_\alpha(\mathbb{r})$ of the harmonic fields in dispersive media by the equation:

$$E^{+}_{\alpha}(\mathbb{r}) = \int d\mathbb{r}' S_{\beta}(\mathbb{r}') G_{\beta\alpha}(\mathbb{r}'|\mathbb{r}). \tag{4}$$

Here $S_{\beta}(\mathbb{r}')$ means the amount of the interactions betwen the field detector located at $\mathbb{r}'$ and the β -component of the electric field on that point. The reaction defined above represents the amount of the interactions of a unit point current source located at $\mathbb{r}$ with the field detectors at some other points.

Now we derive the equations of the reaction field $\mathbb{E}^{+}(\mathbb{r})$. From the field equations (1) and (2), we have

$$\nabla \times \nabla \times \mathbb{E}(\mathbb{r}) - \omega^2 \mu_o (\varepsilon_{\alpha\beta}) \mathbb{E}(\mathbb{r}) = -j\omega\mu_o \mathbb{J}, \tag{5}$$

Introducing the spatial Fourier transform,

$$\mathbb{E}(\mathbb{r}) = \frac{1}{(2\pi)^3} \int d\mathbb{k} \; exp(-j\mathbb{k}\cdot\mathbb{r}) \, \mathbb{E}(\mathbb{k}), \tag{6}$$

into eq.(5), we have after simple algebraic manipulation,

$$E_{\beta}(\mathbb{k}) = -j\omega\mu_o (D)^{-1}_{\beta\alpha} \, J_{\alpha}(\mathbb{k}), \tag{7}$$

where $(D)^{-1}_{\beta\alpha}$ is the (β, α) component of the inverse of the matrix (D) having the matrix element:

$$D_{\beta\alpha} = k^2 \delta_{\alpha\beta} - k_{\alpha} k_{\beta} - \omega^2 \mu_o \varepsilon_{\alpha\beta}. \tag{8}$$

Substituting the unit point current source of the form,

$$\mathbb{J}(\mathbb{r}) = \hat{\mathbb{J}} \, \delta(\mathbb{r}-\mathbb{r}') = \hat{\mathbb{J}} \frac{1}{(2\pi)^3} \int d\mathbb{k} \; exp(-j\mathbb{k}\cdot(\mathbb{r}-\mathbb{r}')), \tag{9}$$

into eq.(7) and using the inverse Fourier transform of $\mathbb{E}(\mathbb{k})$, we have the Fourier integral expression for the dyadic Green function,

$$G_{\beta\alpha}(\mathbb{r}|\mathbb{r}') = -j\omega\mu_o \int d\mathbb{k} (D)^{-1}_{\beta\alpha} exp(-j\mathbb{k}\cdot(\mathbb{r}-\mathbb{r}')). \tag{10}$$

Using the above with eq.(4), we have

$$E^{+}_{\alpha}(\mathbb{r}) = \int d\mathbb{k} \; S_{\beta}(\mathbb{k}) (-j\omega\mu_o (D)^{-1}_{\beta\alpha} exp(-j\mathbb{k}\cdot\mathbb{r})), \tag{11}$$

where we used the relation $D_{\beta\alpha}(\mathbb{k}) = D_{\beta\alpha}(-\mathbb{k})$ which is easily seen from eq.(8). The equation (11) shows that the Fourier component of $E^{+}_{\alpha}(\mathbb{r})$ is given by

$$E^{+}_{\alpha}(\mathbb{k}) = -j\omega\mu_o S_{\beta}(\mathbb{k}) (D)^{-1}_{\beta\alpha}. \tag{12}$$

Comparing eq.(7) with eq.(12) and noting that the electric field $E_{\alpha}(\mathbb{r})$ obeys eq.(5), we find that the reaction field $E^{+}_{\alpha}(\mathbb{r})$ satisfies

$$\nabla \times \nabla \times \mathbb{E}^{+}(\mathbb{r}) - \omega^2 \mu_o (\varepsilon_{\alpha\beta})^T \mathbb{E}^{+}(\mathbb{r}) = -j\omega\mu_o \mathbb{S}(\mathbb{r}), \tag{13}$$

where $(\varepsilon_{\alpha\beta})^T$ means the transposed matrix of $(\varepsilon_{\alpha\beta})$.

Introducing the "magnetic reaction" $\mathbb{H}^{+}(\mathbb{r})$ through

$$\nabla \times \mathbb{E}^{+}(\mathbb{r}) = -j\omega\mu_o \mathbb{H}^{+}(\mathbb{r}), \tag{14}$$

into eq.(13), we obtain

$$\nabla \times \mathbb{H}^{+}(\mathbf{r}) = j\omega\,(\varepsilon_{\alpha\beta})^{T}\,\mathbb{E}^{+}(\mathbf{r}) + \mathbb{S}(\mathbf{r}). \tag{15}$$

It is shown that these equations (14) and (15) of the "reaction fields" take the adjoint forms of the field equations (1) and (2). In other words, the adjoint field equations (14) and (15) are found to describe the physical observable of the "reaction field" defined by eqs.(4) and (14).

3. Reciprocity theorem

Following the usual procedure,[2)] we have the reciprocity theorem from eqs.(1), (2), (14) and (15):

$$\int_S (\mathbb{E}\times\mathbb{H}^{+} - \mathbb{E}^{+}\times\mathbb{H})\cdot n = \int_V d\mathbf{r}\,(\mathbb{J}\cdot\mathbb{E}^{+} - \mathbb{S}\cdot\mathbb{E}), \tag{16}$$

where S denotes the surface of a given volume V, n is outward unit vector normal to S. Here, on the right hand side, the first term $\int d\mathbf{r}\,\mathbb{J}\cdot\mathbb{E}^{+}$ represents the total amount of the "reaction field" produced by the current distribution $\mathbb{J}(\mathbf{r})$ in the volume V, the second term $\int d\mathbf{r}\,\mathbb{S}\cdot\mathbb{E}$ is a part of the amount of the "reaction field" which is dissipated on the detectors contained in the volume V, and the left-hand side is interpreted as the outward flow of the "reaction field" from the surface S. In the limit of infinite volume, the surface integral given by the left hand side of eq.(16) vanilshes because of the dissipations of the medium, so that we have

$$\int d\mathbf{r}\,\mathbb{S}\cdot\mathbb{E} = \int d\mathbf{r}\,\mathbb{J}\cdot\mathbb{E}^{+}. \tag{17}$$

Here the left hand side represents the whole amount of the interactions of the electric field $\mathbb{E}(\mathbf{r})$ with the field detector $\mathbb{S}(\mathbf{r})$ and on the right hand side, it is equivalently evaluated at the position of the field source $\mathbb{J}(\mathbf{r})$ with the use of the concept of "reaction field" $\mathbb{E}^{+}(\mathbf{r})$. The point worthy of note here is that in the reciprocity theorem already derived, $\mathbb{E}^{+}(\mathbf{r})$ and $\mathbb{H}^{+}(\mathbf{r})$ have been considered as the electromagnetic field in another dispersive medium having transposed permittivity tensor $(\varepsilon_{\alpha\beta})^{T}$, whereas, in the above reciprocity theorem they represent the physical observable called "reaction field ", which is defined in the same medium as the adjoint field to the electromagnetic field .

4. Variational Formulae

First we consider the quantity L defined by

$$L \equiv \int d\mathbf{r}\,\mathbb{S}\cdot\mathbb{E} + \int d\mathbf{r}\,[\,\mathbb{E}^{+}(\mathbf{r})\cdot(-\nabla\times\mathbb{H} + j\omega(\varepsilon_{\alpha\beta})\mathbb{E} + \mathbb{J}) + \mathbb{H}^{+}(\mathbf{r})\cdot(\nabla\times\mathbb{E} + j\omega\mu_0\mathbb{H})\,] \tag{18}$$

The first variation of L for the small variations in the field $\mathbb{E}$ is given by

$$\delta_{\mathbb{E}} L = \int d\mathbf{r}\,\delta\mathbb{E}\cdot[\,\mathbb{S} - \nabla\times\mathbb{H}^{+} + j\omega(\varepsilon_{\alpha\beta})^{T}\mathbb{E}^{+}\,], \tag{19}$$

where the partial integration was done under the fixed boundary condition. The stationary condition $\delta_{\mathbb{E}}L = 0$ gives rise to eq.(15). The equation (14) is also obtained from the stationary property of the first variation of L with respect to $\mathbb{H}$. Similarly the small variations of $\mathbb{E}^{+}$ and $\mathbb{H}^{+}$ in the quantity L bring about the electromagnetic fields equations (1) and (2). The value of L under the stationary condition becomes

$$(L)_{st} = \int d\mathbf{r}\, \mathbb{S}\cdot\mathbb{E} = \int d\mathbf{r}\, \mathbb{J}\cdot\mathbb{E}^{+}, \tag{20}$$

which represents the total amount of the interactions of the field source with the field detector $\mathbb{S}(\mathbf{r})$.

The other possible Lagrangian is of fractional type,

$$L' = \frac{\int d\mathbf{r}\, \mathbb{S}\cdot\mathbb{E} \quad \int d\mathbf{r}\, \mathbb{J}\cdot\mathbb{E}^{+}}{\int d\mathbf{r}\left[\mathbb{H}^{+}\cdot(\nabla\times\mathbb{E} + j\omega\mu_0\mathbb{H}) + \mathbb{E}^{+}\cdot(\nabla\times\mathbb{H} - j\omega(\varepsilon_{\theta})\mathbb{E})\right]} \quad . \tag{21}$$

It is shown that the first variation of L' for the small variations in $\mathbb{E}$, $\mathbb{H}$, $\mathbb{E}^{+}$ and $\mathbb{H}^{+}$ vanishes, when these quantities are subject to the field equations (1), (2), (14) and (15).

In these variational formulations (18) and (21), the field source $\mathbb{J}(\mathbf{r})$ and the detector source $\mathbb{S}(\mathbf{r})$ were both fixed. Whereas, in the following variational expression for integral equation, these quantities are varied under the condition that $\mathbb{E}(\mathbf{r})$ and $\mathbb{H}(\mathbf{r})$ are both fixed.

$$L'' = -\frac{\frac{1}{4}\left[\int d\mathbf{r}\,(\mathbb{J}\cdot\mathbb{E}^{+} + \mathbb{S}\cdot\mathbb{E})\right]^2}{\int d\mathbf{r}\, d\mathbf{r}'\left[\mathbb{S}(\mathbf{r})\,\mathbb{G}_T(\mathbf{r}|\mathbf{r}')\,\mathbb{J}(\mathbf{r}')\right]} \tag{22}$$

It is also shown that the first variation of L'' for the small variations in $\mathbb{J}(\mathbf{r})$ and $\mathbb{S}(\mathbf{r})$ vanishes when these quantities satisfy the integral equations (3) and (4). The variational formula (22) derived with the use of the concept of the reaction field is a generalization of the usual stationary expression in the case of vacuum electromagnetic fields[1] which is given formally by putting $\mathbb{E} = \mathbb{E}^{+}$ and $\mathbb{J} = \mathbb{S}$ in eq.(22). The variational expression (22) is applicable to the electromagnetic fields in anisotropic dissipative medium.

References

1) V.H.Rumsey Phys. Rev. 94, 6 (1954) 1483.
2) "Field Theory of Guided Waves" by R.E.Collin chap.1. p.39. Mcgraw-Hill Newyork. Inc. 1960.