A Novel Approach to Remove Singularity of Line-Surface Integral in Higher-Order Algorithms on Triangular Mesh

Fu-Gang Hu[1], Chao-Fu Wang[1], Yeow-Beng Gan[1], and Le-Wei Li [2]

[1] *Temasek Laboratories*
National University of Singapore
5 Sports Drive 2, Singapore 117508

[2] *Department of Electrical and Computer Engineering*
National University of Singapore
Kent Ridge, Singapore 119260

Abstract

Integration of line-surface integral is one of the key issues in modelling scattering and radiation from 3D general bodies using higher-order methods, such as the method of moments (MoM) and finite element-boundary integral (FE-BI) method, with combined field integral equation (CFIE). To accurately evaluate this integration, a novel approach is proposed to analytically remove the singularity of line-surface integral. After removing the singularity, the integration of the line-surface integral can readily be calculated using Gaussian-Legendre quadrature rules. Numerical results demonstrate the proposed approach is able to achieve good accuracy.

1. INTRODUCTION

As is well known, the method of moments (MoM) and finite element-boundary integral (FE-BI) method with combined field integral equation (CFIE) are capable of effective solution of problems of scattering and radiation from 3-D general bodies [1]-[3]. A common issue encountered in these two methods is on the accurate evaluation of the matrix elements which are related to Green's function in free space and its gradient. Due to the strong singularity of the gradient of Green's function, it is difficult to obtain accurate results on integral directly. In order to weaken the singularity of integrand, the gradient operator is often transferred from the Green's function to the test function by using the surface divergence theorem. Accordingly, the original integral is transformed into a new integral which includes two parts: double-surface and line-surface integrals. Hence, only the free space Green's function is involved in both the double-surface and line-surface integrals.

For the integration of the double-surface integral on the coincidental triangles, two approaches can be applied to accurately compute its value by removing the singularity of integral [3]-[6]. In the first approach, Gaussian quadrature rule is used in the first triangle. For each integration point, the second triangle is decomposed into three sub-triangles using the integration point as a vertex in each triangle, followed by the Duffy transformation to remove the singularity. The second approach [4]-[6] provides a way to analytically remove the singularity in the integrand.

Similarly, there are two approaches to calculate the line-surface integral. The first approach uses Gaussian-Legendre quadrature rule in each edge of triangle. For each integration point, the triangle is decomposed into two sub-triangles, and is also followed by the Duffy transformation to remove the singularity.

In this paper, the authors propose a novel approach to analytically remove the singularity in line-surface integral via an adaptation of the idea in [4]-[6]. This new approach can guarantee the accuracy of the calculated matrix element involved in CFIE with higher-order vector basis functions. The combination of this approach with the method discussed in [4]-[6] for double-surface integral forms a complete technique to analytically remove all singularities in CFIE-based MoM and FE-BI method. It should be noted that the method in [4] can only calculate the double-surface integrals in Galerkin electric field integral equation (EFIE) solutions.

2. FORMULATION

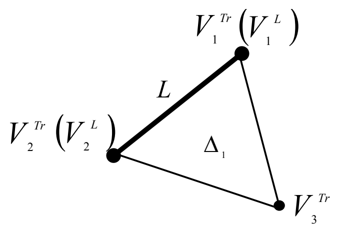


Fig. 1. Relationship of line and triangle in the line-surface integral.

Without loss of generality, assume that the two vertices (v_1^L, v_2^L) in L is coincident with (v_1^{Tr}, v_2^{Tr}) in triangle Δ_1, as shown in Fig. 1. In general, the integral can be written as

$$I = \int_{\xi=0}^{1} \int_{\eta_1=0}^{1} \int_{\eta_2=0}^{1-\eta_1} G \, d\eta_2 \, d\eta_1 \, d\xi \tag{1}$$

where

$$G = \frac{f(\xi,\eta_1,\eta_2)}{\left|\vec{r} - \vec{r}\,'\right|},$$

$$\vec{r}(\xi) = \xi\,\vec{r}_1 + (1-\xi)\,\vec{r}_2, \text{ and}$$

$$\vec{r}\,' = \eta_1\,\vec{r}_1 + \eta_2\,\vec{r}_2 + (1-\eta_1-\eta_2)\,\vec{r}_3 .$$

The relative coordinates are introduced by the following transformations

$$\begin{cases} u_1 = \xi - \eta_1 \\ u_2 = 1 - \xi - \eta_2 \end{cases} \tag{2}$$

Hence, the original integral is changed into

$$I = \int_{\xi=0}^{1} \int_{u_1=\xi-1}^{\xi} \int_{u_2=-u_1}^{1-\xi} G \, du_2 \, du_1 \, d\xi \tag{3}$$

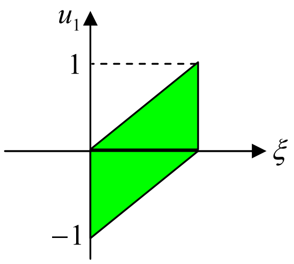


Fig. 2. Integration domain about ξ and u_1.

Decomposing the integration domain about ξ and u_1 (see Fig. 2) into two sub-domains and interchanging the order of integration of ξ and u_1 yield

$$I = \int_{u_1=-1}^{0} \int_{\xi=0}^{u_1+1} \int_{u_2=-u_1}^{1-\xi} G \, du_2 \, d\xi \, du_1 + \int_{u_1=-1}^{1} \int_{\xi=u_1}^{1} \int_{u_2=-u_1}^{1-\xi} G \, du_2 \, d\xi \, du_1 \tag{4}$$

Applying the same technique to (4) with consideration of the integration domain about ξ and u_2 as shown in Fig. 3(a) and (b) gives us

$$I = \int_{u_1=-1}^{0} \int_{u_2=-u_1}^{1} \int_{\xi=0}^{1-u_2} G \, d\xi \, du_2 \, du_1 + \int_{u_1=0}^{1} \int_{u_2=-u_1}^{0} \int_{\xi=u_1}^{1} G \, d\xi \, du_2 \, du_1 + \int_{u_1=0}^{1} \int_{u_2=0}^{1-u_1} \int_{\xi=u_1}^{1-u_2} G \, d\xi \, du_2 \, du_1 \tag{5}$$

Substituting $z_1 = -u_1$, $z_2 = -u_2$ and $z_3 = u_1 + u_2$ into the three parts of (5), respectively, and interchanging the order of integration of (u_2, z_1) and (u_1, z_3), one can write (5) as

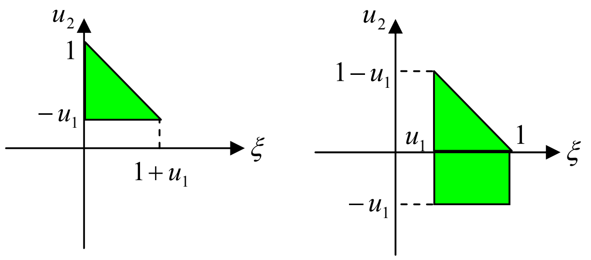
$$I = I_1 + I_2 + I_3 \tag{6}$$

where

$$I_1 = \int_{u_2=0}^{1} \int_{z_1=0}^{u_2} \int_{\xi=0}^{1-u_2} G \, d\xi \, dz_1 \, du_2 \tag{7}$$

$$I_2 = \int_{u_1=0}^{1} \int_{z_2=0}^{u_1} \int_{\xi=u_1}^{1} G \, d\xi \, dz_2 \, du_2 \tag{8}$$

$$I_3 = \int_{z_3=0}^{1} \int_{u_1=0}^{z_3} \int_{\xi=u_1}^{1-u_2} G \, d\xi \, du_1 dz_3 \tag{9}$$



(a) (b)

Fig. 3. Integration domain about ξ and u_2.

Applying Duffy transformation to the first two variables in (7)-(9) gives the following new formulation in which the singularity is removed.

$$I = \sum_{i=1}^{3} I_i , \tag{10}$$

where

$$I_i = \int_{\omega=0}^{1} \int_{x=0}^{1} \int_{\xi=L_i}^{U_i} \omega \, G(\xi,\eta_1,\eta_2) \, d\xi \, d\eta_1 \, d\eta_2$$

i=1	$L_i = 0$, $U_i = 1-\omega$	$\eta_1 = \xi + \omega x$, $\eta_2 = 1 - \xi - \omega$
i=2	$L_i = \omega$, $U_i = 1$	$\eta_1 = \xi - \omega$, $\eta_2 = 1 - \xi + \omega x$
i=3	$L_i = \omega x$, $U_i = 1-\omega(1-x)$	$\eta_1 = \xi - \omega x$, $\eta_2 = 1 - \xi - \omega(1-x)$

The integration of I as given in (10) can be accurately performed using the Gaussian-Legendre quadrature rule.

3. NUMERICAL RESULTS

As an example, an integral with available analytical result is chosen to verify the accuracy of the proposed method.

The integrand of integral (1) is chosen as

$$G = f(\xi,\eta_1,\eta_2)/\left|\vec{r}-\vec{r}'\right|$$

where $f(\xi,\eta_1,\eta_2) = 5(\xi-1)[6(1-\eta_2)-3\xi_1](1-\eta_1-\eta_2)$ is a higher-order polynomial. The region of integration chosen is a triangle with vertices

$$\begin{cases} \vec{r}_1 = (1.0, 0.0, 0.0) \\ \vec{r}_2 = (0.0, 0.0, 0.0) \\ \vec{r}_3 = (1.0, 1.0, 0.0) \end{cases}$$

The analytical result of the integral in the test case is $I_0 = 3.5 - 2^{2.5}$.

To verify the accuracy of the numerical integration method proposed, the relative error is defined as $\varepsilon = \left|(I - I_0)/I_0\right|$. Figure 4 shows the relative error versus the number of integration points. It is observed that the proposed method can achieve very good accuracy beyond a certain number of integration points. The monotonous decrease in the relative error shows the proposed method is capable of removing the singularity in line-surface integral.

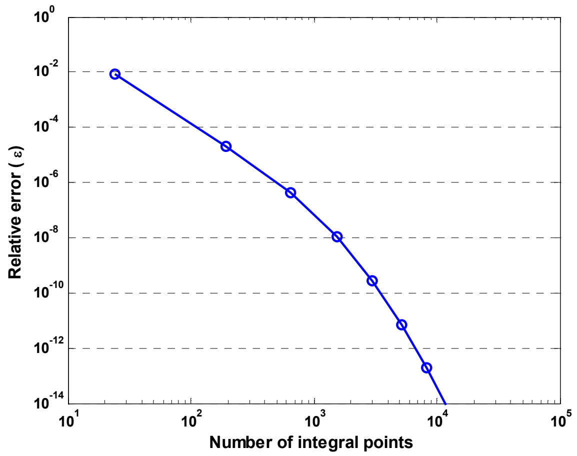


Fig. 4. The relative error versus number of integration points.

4. CONCLUSION

The proposed approach is able to analytically remove the singularity in line-surface integrals, which facilitates accurate computations of the matrix element involved in the MoM and FE-BI method with CFIE and higher-order vector basis function.

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