

# Progress in ADI-FDTD method

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## Abstract

*Due to unconditional stability of ADI-FDTD, it is becoming popular. In this paper progress of the ADI-FDTD method is given. First of all to get the advantages of both ADI-FDTD and FDTD methods a hybrid method is briefly explained. Then to reduce the error of the ADI-FDTD method at larger time steps error reduced ADI-FDTD is presented. To control the dispersion of the conventional ADI-FDTD method at higher time steps dispersion optimized ADI-FDTD method with different combinations for optimization is given.*

## I Introduction

With the development of more complicated structures in the field of electrical engineering, the importance of the computational electromagnetics increases. With computational electromagnetics, a designer can know what happens inside circuit structures, namely, which components or elements radiate and how signals travel and reflect. For microwave frequencies and wide band system applications, time-domain methods are increasingly preferred of their capability in handling wide band signals. Among time domain methods, the FDTD (Finite Difference Time Domain) method [1] has attracted more attention due to its simplicity and direct applicability to Maxwell's equations. It has been used in a large number of applications, and FDTD based software has been developed commercially. Nevertheless, due to CFL (Courant-Friedrich-Levy) stability constraint and numerical dispersion error, it takes large memory and simulation time for electrically large and high Q structures. To circumvent the problems, many improved FDTD methods have been developed. For instance, to make FDTD method memory efficient, PSTD (Pseudospectral Time Domain) method [2] was proposed, and to reduce its dispersive error MRTD (Multiresolution Time Domain) method [3] was developed. To remove the CFL constraint, unconditionally stable ADI-FDTD (Alternating Direction Implicit FDTD) method [4-5] was introduced recently, although it takes more memory and is more dispersive at larger time steps.

To get the advantages and improve the efficiencies of both FDTD and ADI-FDTD methods, a hybrid FDTD and ADI-FDTD method is discussed in section II. The ADI-FDTD method has higher error at higher time steps, to reduce it in section III two different error-reduced ADI-FDTD methods are presented. These error-reduced methods are based on the more accurate Crank Nicolson (CN) method but the simulation procedure is like the ADI-FDTD method.

To reduce the dispersion of the ADI-FDTD method at higher time steps, in section IV, dispersion improvement for the ADI-FDTD method is discussed, in section V numerical results and at the end conclusions are given.

## II Hybrid FDTD/ADI-FDTD method

In FDTD the problem of large memory and long CPU time arises when a structure that contains electrically small geometric features, including sharp conducting edges, is modeled. To have the highest possible computational efficiency, efficient techniques have been proposed that incorporate *a priori* knowledge of the field behaviors near these fine structures into the FDTD algorithms with little increase of computational expenditures [6]. However, they require the field behaviors to be known and analytical processing to be done beforehand. Alternatively, the so-called subgridding schemes [7] can be used. In these subgridding schemes, fine numerical grids or meshes are applied to fine geometric features where strong field variations occur. In the rest of the solution domain, coarse grids or meshes are still used in order to minimize the memory usage. A scheme is required to interface the fine grid and the coarse grid, both in space and in time, since the fine grids possess different spatial and temporal properties and conditions.

In the subgridding scheme based purely on the conventional FDTD, one of the critical issues is to interface the two grids in time. In a fine grid, the time step has to be small, as stipulated by the CFL condition due to the fine grid cell size. However, in the coarse grid, the time step can be much larger. The result is that the two grids are simulated with two different time steps, or they simply are not synchronized. Consequently, a very carefully designed scheme is needed to interface the two grids, not only in space, but also in time where they are joined. Past experience has proven that unless the small time step for the fine grid is also applied to the coarse grid, an interfacing scheme often leads to instability or an overly complicated technique, which may still have a late-time stability problem [8].

To make FDTD more efficient, hybrid schemes have been proposed in the past. Those were aimed at taking advantage of the methods that are efficient with different conditions. For instance, the FDTD method has been interfaced with many other methods such as TLM that has less numerical dispersion and MoM that is effective for modeling open regions [9-10]. To save memory and computation time, we proposed a hybrid FDTD and ADI-FDTD technique. The details can be found in

[11-13]. In it, the ADI-FDTD is applied to fine mesh and the FDTD to coarse mesh. A subgridding interface is then developed and the associated interpolation is applied in space domain only. This technique is efficient for the applications in which there is need of fine subgrids. It leads to savings in memory and computation time.

The reason to make the hybrid FDTD and ADI-FDTD is that, for applications where fine meshes are applied, the FDTD method requires long simulation time due to the small time step stipulated by the FDTD inherent CFL stability condition. To tackle this problem, the unconditionally stable ADI-FDTD method can be applied since it does not have the CFL stability condition. However, the ADI-FDTD normally requires more memory than the FDTD method. Therefore, it will be efficient not to apply the ADI-FDTD but the FDTD to wherever fine meshes are not needed. This leads to a subgridding scheme where an interfacing algorithm is needed to connect the FDTD-applied coarse mesh and the ADI-FDTD-applied fine mesh. In such a way, advantages of both FDTD and ADI-FDTD methods are utilized and the usage of the computation resources are optimized.

### The Hybrid Method Interfacing

For hybrid method with a general cell ratio of  $m$ , the intermediate points at interface can be easily found with:

$$h_l = \frac{m-l}{m} H_z|_A + \frac{l}{m} H_z|_B \quad (1)$$

where  $h_l, l=1,2,\dots,m-1$ , are the field values to be interpolated in between points  $A$  and  $B$ . It should be pointed out here that the relatively large time-step applied in the dense grid will not cause unacceptable numerical dispersion errors as long as the time step does not cause unacceptable errors in the coarse grid. This occurs because the ADI-FDTD and FDTD present similar dispersion errors if their time and spatial step sizes are comparable. In our case, the ADI-FDTD actually has a smaller grid size than the FDTD. Therefore, the numerical dispersion error in the dense grid should not be larger than the coarse grid.

### III Error Reduced ADI-FDTD Methods

Unconditionally stable ADI-FDTD and Crank-Nicolson (CN) FDTD methods have been proposed recently [4-5][16]. Due to their unconditional stability, both methods have attracted much attention in recent years. It has been found that although the ADI-FDTD is computationally efficient, but it has large errors with large time steps. Such a property compromises the uses of the ADI-FDTD method with its unconditional stability. On the other hand, the CN-FDTD method is of much high accuracy, even with large time steps, but at the cost of much larger computational time [17]. Therefore, it will be desirable to develop a method that has the advantages of both methods. In fact, further studies on the ADI-FDTD and the CN-FDTD

methods have shown that the ADI-FDTD method can be considered as the perturbed form of the CN method with the so-called splitting error term [16]. Based on it, an iterative method that solves the CN-FDTD method in an ADI-FDTD fashion was reported [18]. It embodies a loop of iterations at each FDTD marching time step. Consequently, it achieved much higher accuracy than that of the ADI-FDTD at the cost of more computation time. In this section, two novel ADI-FDTD methods are proposed that are based on the CN-FDTD formulation but with the same computational efficiency as that of the conventional ADI-FDTD. Although the achieved accuracy may not be as high as that with the CN-FDTD method, it is sufficient for most of the large time step simulations without increasing computation expenditures. We name the methods as the error-reduced (ER) ADI-FDTD methods. Further detail can be found in [19]

### Formulations of the 2-D ER-ADI-FDTD

#### Method #1

For this method, the two ADI-FDTD steps are computed as:

#### Step 1

$$([I] - \frac{\Delta t}{2}[A])U^{n+\frac{1}{2}} = ([I] + \frac{\Delta t}{2}[B])U^n + \frac{\Delta t^2}{4}[A][B](U^n - U^{n-\frac{1}{2}})$$

and

#### Step 2

$$([I] - \frac{\Delta t}{2}[B])U^{n+1} = ([I] + \frac{\Delta t}{2}[A])U^{n+\frac{1}{2}} + \frac{\Delta t^2}{4}[A][B](U^{n+\frac{1}{2}} - U^n)$$

#### Method #2

In this method, following two-steps are computed:

#### Step 1

$$([I] - \frac{\Delta t}{2}[A])U^{n+1} = ([I] + \frac{\Delta t}{2}[B])U^n + \frac{\Delta t^2}{8}[A][B](U^n - U^{n-1})$$

and

#### Step 2

$$([I] - \frac{\Delta t}{2}[B])U^{n+2} = ([I] + \frac{\Delta t}{2}[A])U^{n+1} + \frac{\Delta t^2}{8}[A][B](U^{n+1} - U^n)$$

The equations above are different from the conventional ADI-FDTD formulations [4-5] in the addition of terms on the right-hand sides that are proportional to the square of the time step [16]. The terms are related to the truncation errors and the additions of these in the computations are therefore expected to reduce the errors.

### IV Dispersion Optimized ADI-FDTD Method

ADI-FDTD faces the problem of larger dispersion error with bigger time steps. To get the real advantage of this unconditional stability, efforts have been put into controlling this dispersion problem [20-22]. In [20], dispersion reduction is presented for the two-dimensional case. In [21], higher order ADI-FDTD is introduced to reduce the dispersion but at

the cost of simulation time and memory. In the previous section, error-reduced terms are added into the existing ADI-FDTD method to get the better results but instability problems still exist for the three-dimensional cases. In [22], improvement in dispersion is presented for the 2D ADI-FDTD method by introducing artificial anisotropy.

Similar to the FDTD method, the dispersion of the ADI-FDTD method is an inherent feature of the algorithm, which affects the overall accuracy. To circumvent this problem, reduction in cell sizes are needed and, as a result, increase the computation load. Here, novel ways to minimize the numerical dispersion of the ADI-FDTD method are presented. In them, additional controlling parameters are introduced to reduce the dispersion error. These dispersion-optimized ADI-FDTD (DO-ADI-FDTD) methods improve accuracy in comparison with the original ADI-FDTD method, without additional computational complexity and loads.

## V Numerical Results

In this section numerical results of the above-described methods are given.

### A Hybrid method

For this method only two applications are presented for explanation. The hybrid scheme described above was applied first to a two-dimensional finned waveguide. Since this structure was also solved in [14] with the other techniques, comparisons and validations of the proposed scheme can be made. The finned waveguide computed in our study has dimensions  $a = 2b = 64\text{mm}$ . The crosssectional view of it is shown in Fig.1. Only one-quarter of the waveguide needs to be modeled (as shown in Fig 2) due to symmetry of structure, where M and E represent magnetic and electric wall respectively. The fin length considered was equal to  $b/4$ . Because of the expected rapid changes of the fields around the fin, a dense mesh is applied in the vicinity of the fin. As described earlier the ADI-FDTD is applied in the fine grid region and the conventional FDTD is applied in the coarse grid region.

Three different cell sizes  $\Delta x' = \Delta y' = \Delta l = .5\text{mm}$ ,  $0.25\text{mm}$ , and  $0.125\text{mm}$  were computed. For reference purposes, a separate pure FDTD simulation was also run with a uniform grid whose cell size was equal to the dense grid cell size of the hybrid method. The reason for taking a uniformly fine FDTD grid is that the FDTD can achieve a level of accuracy similar to that of the proposed method as shown in Tables. The comparison made under such a condition can then be deemed fair. Tables 1 and 2 also show the computer resources used by the proposed hybrid method and FDTD method respectively for comparison purpose. In both cases, the computer used is a Dell XPS T600 with 600 MHz CPU and 256MB of RAM. From tables it is also clear that for both methods, the smaller the cell size, the more accurate the results. The difference of normalized cut-off frequency of both methods decreases with the reduction in cell size and

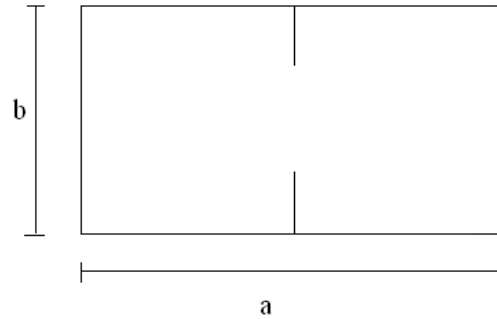


Fig. 1 Cross section of the finned waveguide

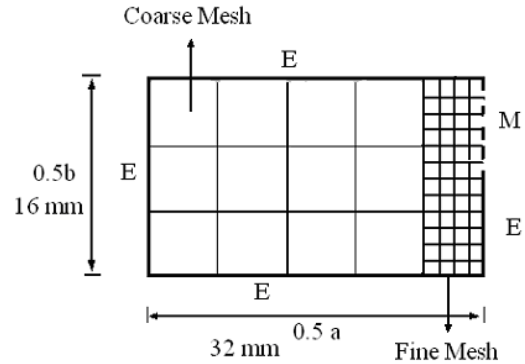


Fig. 2 One quarter of finned waveguide

both methods converge to produce the same solution, as the cell size tends to zero.

Table 1 Computer resources used by the hybrid method for the finned waveguide

| Size of fine mesh cell | Normalized cut-off frequency obtained | Time for simulation (Seconds) | Memory used by program |
|------------------------|---------------------------------------|-------------------------------|------------------------|
| .5mm                   | 0.2241                                | 2                             | 632K                   |
| .25mm                  | 0.2263                                | 13                            | 708K                   |
| .125mm                 | 0.2271                                | 99                            | 992K                   |

Table 2 Computer resources used by the conventional FDTD for the finned waveguide.

| Size of fine mesh cell | Normalized cut-off frequency obtained | Time for simulation (Seconds) | Memory used by program |
|------------------------|---------------------------------------|-------------------------------|------------------------|
| .5mm                   | 0.2249                                | 3                             | 688K                   |
| .25mm                  | 0.2267                                | 20                            | 936K                   |
| .125mm                 | 0.2273                                | 313                           | 1912K                  |

The second application considered was 3 dimensional planar microstrip structure with dimensions 20.32 x 5.565 x 33.864 mm, strip width 0.784mm and distance between strip and ground plate is 2.413mm. Its crosssectional view is shown in Fig. 3.

In this case the cell ratio used for the coarse and fine mesh is 1:2; for the pure FDTD case, the whole domain was modeled with fine mesh. To terminate open solution domains, the convolutional perfectly matched layer (CPML) [15] was implemented. Time used by the hybrid method and FDTD method is 553 and 1227 seconds respectively.

The effective dielectric constant used for this structure is shown in Fig.4. It can be observed that there is a very good agreement between the proposed hybrid method and the FDTD method. In all, it is concluded that the proposed hybrid method is effective and efficient for numerical subgridding.

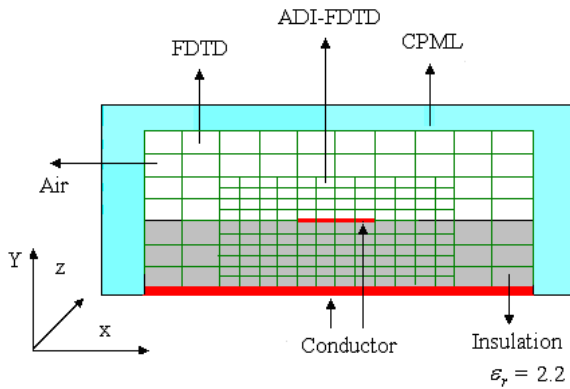


Fig. 3 Cross sectional view of microstrip line

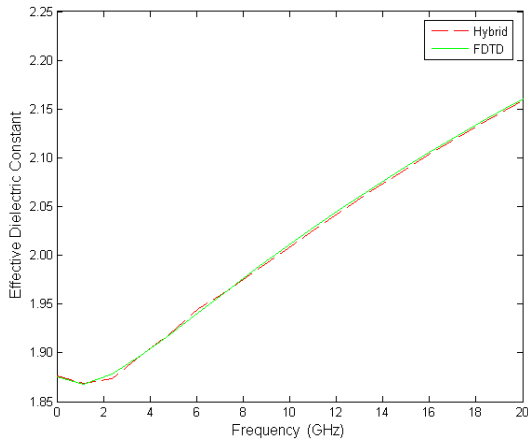


Fig.4 Effective dielectric constant vs. frequency for the microstrip line

### B Error Reduced ADI-FDTD method

In this subsection, proposed Method # 1 and Method # 2 are simulated, and their results are compared with the conventional ADI-FDTD and FDTD methods. A structure of two parallel plates of zero thickness in free space was

considered and studied. The plates were 2 m long and have a distance of 0.2 m in between them. To truncate the surrounding environment for simulation purpose, Perfect Magnetic Wall (PMW) was used on all four sides. Cell size considered for both methods in each direction was 0.2 meter. The raised cosine waveform with frequency 750 kHz was used as a source. Fig. 5 shows the computed electric field  $E_y$  along the  $x$  axis with different CFL factor “ $s$ ” which is the ratio of time step to the CFL limit. It can be observed from this figure that for  $s \leq 15$  results with the Method # 1 are approximately the same as those with the conventional ADI-FDTD method at  $s = 0.5$ , while the results with Method # 2 are similar but have much larger errors. As can be seen, all the methods have similar errors at  $s \leq 0.5$ . Simulation time for both methods was same as for the conventional ADI-FDTD method with minor increment in memory for proposed methods. The memories used with three methods are 1.336MB (with the conventional ADI-FDTD), 1.344 MB (with Method #1) and 1.344 MB (with Method #2). The proposed two methods used slightly more memory but insignificantly.

Three dimensional error reduced ADI-FDTD method was found to be unstable to-date, Efforts are continued to find some ways to make 3-D ER-ADI-FDTD methods stable, so that the efficiency of the method can be better than the conventional ADI-FDTD method and faster than the CN method in 3D case.

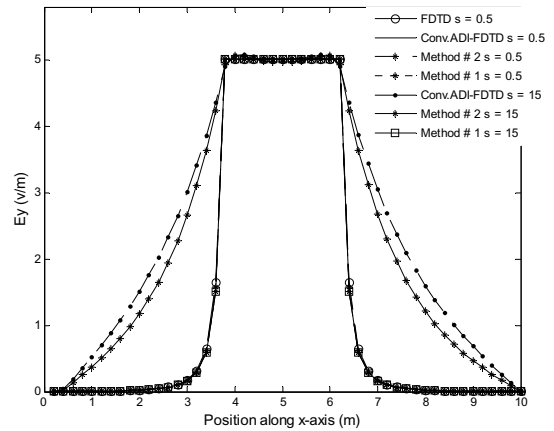


Fig. 5 Electric field  $E_y$  for FDTD, ADI-FDTD and proposed methods

### C 3-D Dispersion Optimized -ADI-FDTD

The dispersion equation for 3-D dispersion optimized ADI-FDTD method is:

$$\sin^2\left(\frac{\omega\Delta t}{2}\right) = \frac{\mu\epsilon(\mu\epsilon P_x^2 + \mu\epsilon P_y^2 + \mu\epsilon P_z^2 + P_x^2 P_y^2 + P_y^2 P_z^2 + P_z^2 P_x^2)}{(\mu\epsilon + P_x^2)(\mu\epsilon + P_y^2)(\mu\epsilon + P_z^2)} \quad (2)$$

$$P_x = B \frac{\Delta t}{\Delta x} \sin\left(\frac{k_x \Delta x}{2}\right), P_y = A \frac{\Delta t}{\Delta y} \sin\left(\frac{k_y \Delta y}{2}\right), P_z = C \frac{\Delta t}{\Delta z} \sin\left(\frac{k_z \Delta z}{2}\right)$$

where  $k_x$ ,  $k_y$  and  $k_z$  are wavenumbers in  $x$ ,  $y$  and  $z$  directions respectively. In the three-dimensional case, three controlling parameters  $A$ ,  $B$  and  $C$  are introduced. Different combinations of  $A$ ,  $B$  and  $C$  will lead to different dispersion characteristics. In the following, four combinations are considered.

In combination 1,

$$A = \frac{\sqrt{\mu\epsilon} \Delta y \tan\left(\frac{1}{2} c k \Delta t\right)}{\sin\left(\frac{1}{2} k \Delta y\right) \Delta t}, \quad B = \frac{\sqrt{\mu\epsilon} \Delta x \tan\left(\frac{1}{2} c k \Delta t\right)}{\sin\left(\frac{1}{2} k \Delta x\right) \Delta t}$$

$$C = \frac{\sqrt{\mu\epsilon} \Delta z \tan\left(\frac{1}{2} c k \Delta t\right)}{\sin\left(\frac{1}{2} k \Delta z\right) \Delta t}$$

In this combination, the setting is used to see the dispersion characteristics when optimization is made in all three directions ( $x$ ,  $y$  and  $z$ ).  $A$  is obtained by setting  $\theta = 90^\circ$  and  $\phi = 0^\circ$ ,  $B$  is obtained by setting  $\theta = 90^\circ$  and  $\phi = 90^\circ$ ,  $C$  is selected by setting  $\theta = 0^\circ$  and  $\phi = 90^\circ$  in equation (2).

In combination 2,  $A$  and  $B$  are taken as one,  $C$  is same as in combination 1. This combination gives better result than the conventional ADI-FDTD method. In combination 3,  $B$  is same as in combination 1, the parameters  $A$  and  $C$  are set to 1 in order to examine the dispersion characteristics due to the optimization in the  $x$ -direction only. In combination 4,  $A$  is same as in combination 1, the parameters  $B$  and  $C$  are set to 1 in order to examine the dispersion characteristics due to the optimization in the  $y$ -direction only. Fig 6 shows the numerical dispersion for combination 1 with CFL factor equal to 5 and it demonstrates that the DO-ADI-FDTD is better than the conventional method, plots for the rest of combinations and more details can be found in [23].

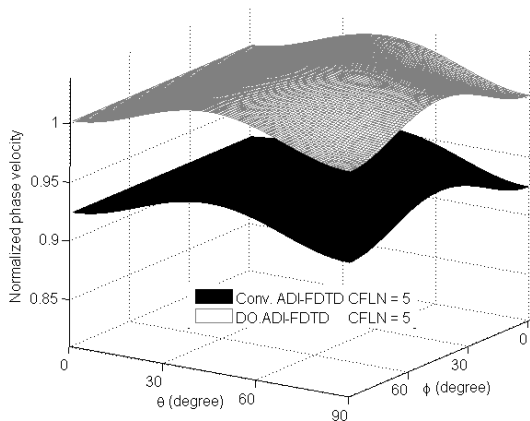


Fig. 6 Numerical dispersion for combination 4 and the conventional ADI-FDTD method

Fig 7 shows the absolute error versus the CFL number. The

DO-ADI-FDTD method shows less dispersion than that of the conventional method.

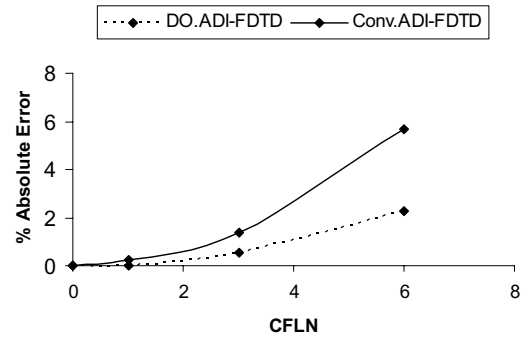


Fig. 7 Absolute error vs. CFLN with combination 3

## VI Conclusions

In summary all methods in this paper are aimed at improving the efficiency of FDTD method and the ADI-FDTD method, especially of the ADI-FDTD method. To improve the computational efficiency an efficient hybrid method is introduced. It is found that the hybrid technique improves the computation efficiency in terms of both CPU time and memory for modeling RF/microwave structures. To have better results with larger time steps, error-reduced ADI-FDTD methods are introduced. They not only retain the same numerical computational efficiency as the conventional ADI-FDTD method, but also achieve similar accuracy to that with the CN-FDTD. In particular, the first method presents superior performance with the large time steps. To control the dispersion of the ADI-FDTD method dispersion optimized ADI-FDTD method is proposed with the introduction of dispersion controlling parameters. Choices of different controlling parameters are investigated. For all the cases proposed methods are in general better than the conventional ADI-FDTD method; however, the controlling parameters need to be chosen optimally to lead to better dispersion error reductions.

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