STABILITY AND DISPERSION ANALYSIS OF A FOURTH-ORDER FDTD SCHEME

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I. Introduction

Yee's finite-difference time-domain (FDTD) method has been widely used for time-domain electromagnetic field simulations [1]-[3]. However, as the size of a structure or its computational domain becomes larger, the length of the time integration has to increase to capture electromagnetic interactions across the electrically large computational domain. Furthermore, the number of grid points per wavelength needed for a given accuracy increases with the length of the integration in time [4], [5]. With second-order accuracy of the standard Yee's FDTD scheme, the computational cost becomes prohibitively large for accurate full-wave time-domain solutions of electrically large structures. To address the limitations of second-order explicit methods, many high-order explicit FDTD schemes have been explored for their numerical efficiency [3], [6]. In [7], a long-time stable fourth-order accurate FDTD scheme was proposed and its fourth-order convergence was demonstrated using a two-dimensional (2-D) partially-filled cavity geometry. This paper presents the proposed FDTD equations in 2-D transverse magnetic (TM) polarization case and investigates its stability condition and numerical errors.

II. Fourth-Order FDTD Equations (2-D TM case)

Maxwell's equations for 2-D TM polarization case is considered here for simplicity. Assuming isotropic, homogeneous, lossless medium, Maxwell's equations can be written as

$$\frac{\partial U}{\partial t} = L(U) \tag{1}$$

where

$$U = \begin{bmatrix} H_x \\ H_y \\ E_z \end{bmatrix}, \; L(U) = \begin{bmatrix} -\frac{1}{\mu}\frac{\partial E_z}{\partial y} \\ \frac{1}{\mu}\frac{\partial E_z}{\partial x} \\ -\frac{1}{\epsilon}\frac{\partial H_x}{\partial y} + \frac{1}{\epsilon}\frac{\partial H_y}{\partial x} \end{bmatrix}. \tag{2}$$

Applying the fourth-order staggered backward differentiation method [8] to (1) for temporal integration,

$$\begin{aligned} U^{n+1} \;=\; & \frac{17}{22}U^n + \frac{9}{22}U^{n-1} - \frac{5}{22}U^{n-2} + \frac{1}{22}U^{n-3} \\ & + \frac{12}{11}\Delta t L(U)^{n+1/2} + O((\Delta t)^5). \end{aligned} \tag{3}$$

For spatial derivatives in $L(U)$, the fourth-order staggered central difference approximation is used as in Fang's (2,4) FDTD scheme [6] and the resulting difference equations are of fourth-order accuracy in space and time. KEES

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III. Formal Analysis and Discussion

For formal analysis of the fourth-order FDTD equations in 2-D TM case, consider a plane wave eigenmode in a 2-D space represented by

$$\begin{bmatrix} H_{x}{}^{n}_{i_x,i_y} \\ H_{y}{}^{n}_{i_x,i_y} \\ E_{z}{}^{n}_{i_x,i_y} \end{bmatrix} = Z^n e^{-j[k_x i_x \Delta x + k_y i_y \Delta y]} \begin{bmatrix} H_{x0} \\ H_{y0} \\ E_{z0} \end{bmatrix} \tag{4}$$

where i_x, i_y, k_x, and k_y denote space indexes and wavenumbers along the x and y directions, respectively. Then, the proposed fourth-order difference equations described in the previous section can be cast in a matrix equation

$$\mathbf{AX} = \mathbf{0}. \tag{5}$$

The Routh-Hurwitz method [9],[10] is used to find the stability condition

$$\Delta t \leq \frac{3}{7}\left[c\sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}}\right]^{-1}. \tag{6}$$

Solving complex polynomial equations associated with the difference equations in (5) leads to the dispersion and dissipation properties of the proposed fourth-order FDTD scheme. Fig.1 shows the roots of the characteristic equation for the fourth-order scheme as a function of spatial resolution kh, where k and h are the wavenumber and the cell size, respectively. The amplitudes of physical modes represented by solid lines become slightly smaller than 1 as kh increases. This indicates that the fourth-order scheme is numerically dissipative. The amplitude errors of the proposed fourth-order FDTD scheme and Yee's second-order scheme are compared in Fig.2. Yee's second-order FDTD scheme is non-dissipative showing no amplitude error while the proposed fourth-order scheme exhibits some amplitude error as the spatial frequency increases. However, the numerical dissipation is negligibly small. For the moderate spatial resolution of $kh = 2\pi/10$, the amplitude error is less than 2×10^{-6}. Fig.3 compares the phase errors of the fourth-order scheme and Yee's second-order scheme. It can be seen that Yee's second-order scheme suffers much more numerical dispersion error than the fourth-order scheme. For $kh = 2\pi/10$, the phase errors of the second-order Yee's scheme and the proposed fourth-order scheme are 0.0068 and 2.2825×10^{-4}, respectively. The superb accuracy of the fourth-order scheme can be clearly seen again in Fig.4 where the phase errors versus propagation angle of the fourth-order and second-order FDTD methods are plotted.

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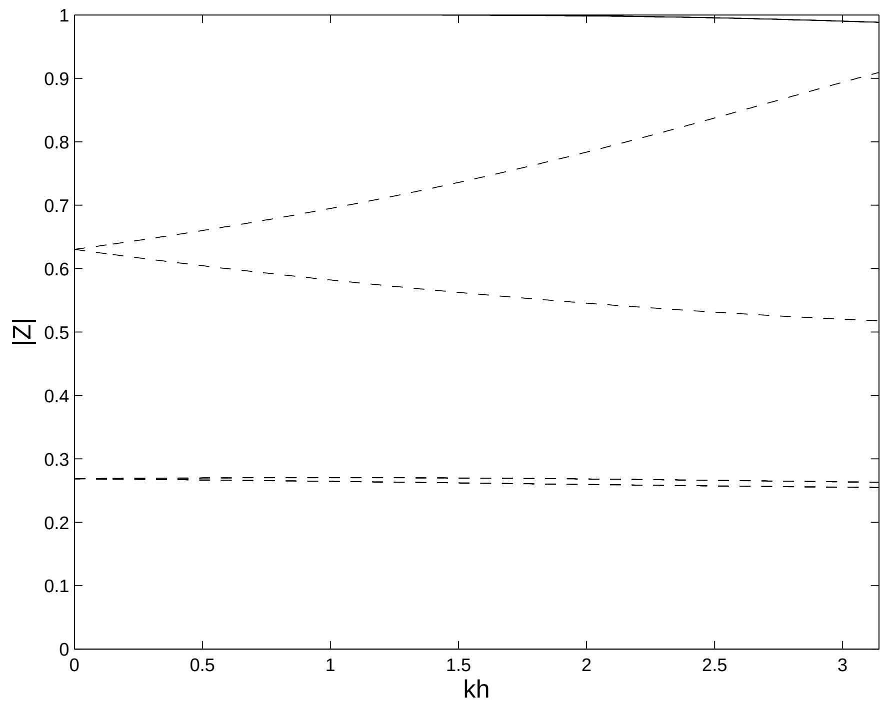

Fig. 1. Magnitude $|Z|$ of eight roots of the characteristic polynomial equation for the fourth-order scheme as a function of kh. The Courant number, $s = 0.3$ and the propagation angle, $\alpha = 45^{\circ}$. The solid and dashed lines represent physical and computational modes, respectively.

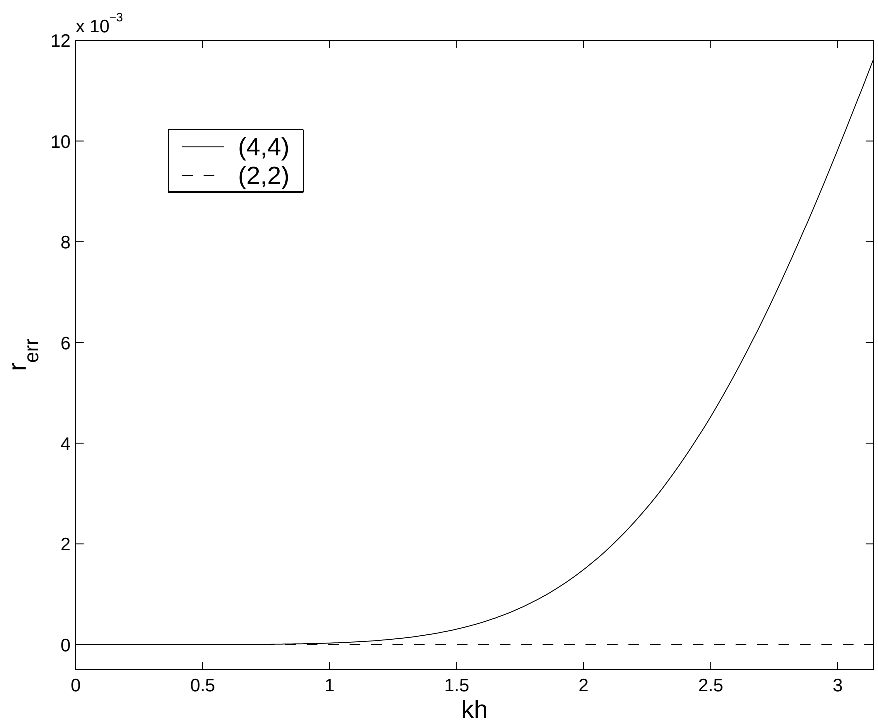

Fig. 2. Relative amplitude errors of the proposed (4,4) scheme and Yee's (2,2) scheme. $s_{4,4} = 0.3$, $s_{2,2} = 0.7$ and $\alpha = 45^{\circ}$.

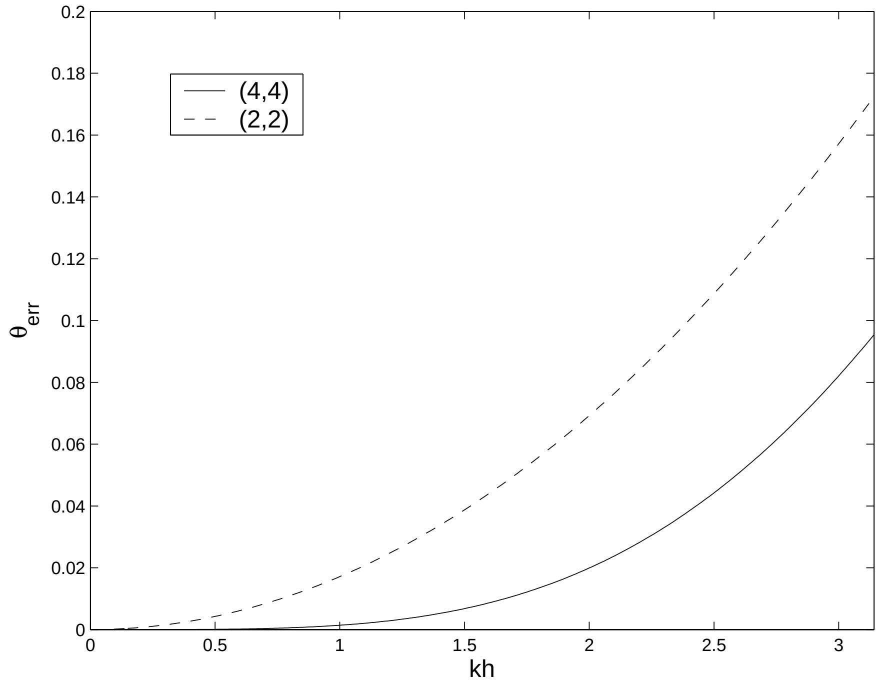

Fig. 3. Relative phase errors of the proposed (4,4) scheme and Yee's (2,2) scheme. $s_{(4,4)} = 0.3$, $s_{(2,2)} = 0.7$ and $\alpha = 45°$.

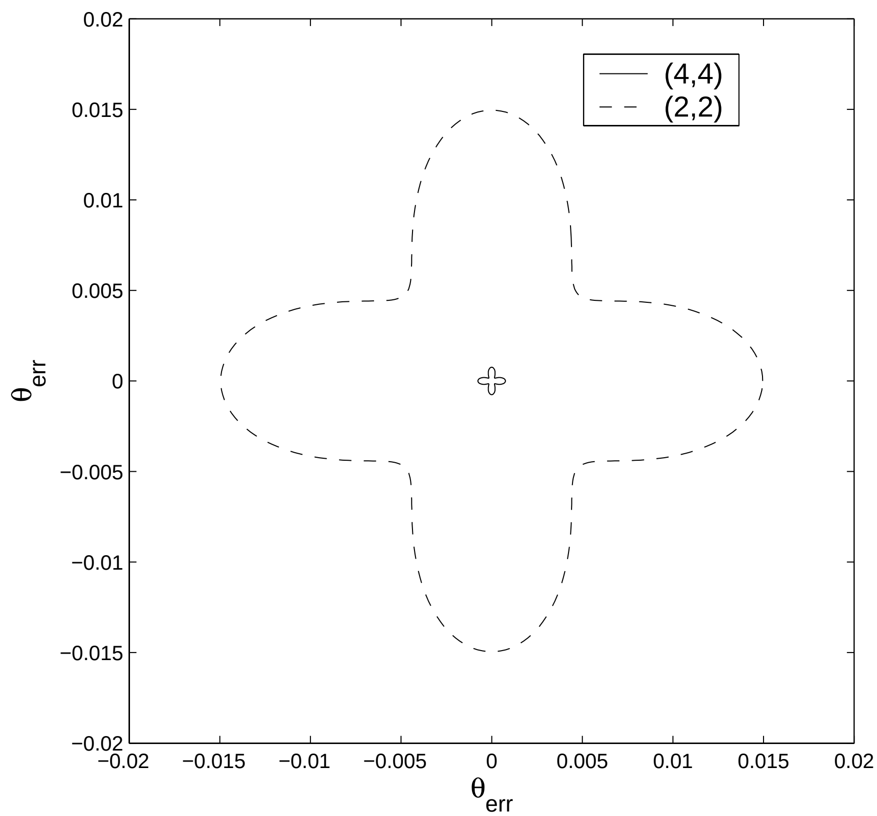

Fig. 4. Relative phase errors versus propagation angle of the proposed (4,4) scheme and Yee's (2,2) scheme. $s_{(4,4)} = 0.3$, $s_{(2,2)} = 0.7$ and $\alpha = 45°$.