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WIENER–HOPF ANALYSIS OF THE HIGH-FREQUENCY DIFFRACTION BY A THIN MATERIAL STRIP

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1. Introduction

Approximate boundary conditions can be widely used to simplify analytical and numerical solutions of scattering problems involving complicated structures. The simplest conditions are the standard impedance boundary condition applicable at the surface of a lossy material and the related transition condition modeling a thin material layer as a current sheet. Thin lossy material layers are of great importance for radar cross section (RCS) and target identification studies. A mathematical model of such a layer is a resistive sheet, and there have been extensive investigations on the scattering by resistive strips [1–3]. However, the solutions become increasingly inaccurate at broad incident angles when the electric vector has a component normal to the layer. Recently it was suggested by Senior and Volakis [4, 5] that a thin material layer can be effectively modeled by a pair of modified resistive and conductive sheets (or resistive and modified conductive sheets) each satisfying given boundary conditions, which are shown to provide accurate modeling of thin material layers arising in diffraction problems [6, 7].

In this paper, we shall consider a thin, homogeneous material strip as an example of geometries modeled by approximate boundary conditions, and analyze the E-polarized plane wave diffraction by means of the Wiener–Hopf technique [8]. Introducing the Fourier transform for the scattered field and applying boundary conditions in the transform domain, the problem is formulated in terms of the simultaneous Wiener–Hopf equations, which are solved rigorously via the factorization and decomposition procedure. However, the solution is formal in the sense that branch-cut integrals with unknown integrands are involved. Approximate methods are further applied for evaluation of the branch-cut integrals, and a high-frequency asymptotic solution, valid for the strip width large compared with the wavelength, is explicitly derived.

The time factor is assumed to be $e^{-i\omega t}$ and suppressed throughout the following analysis.

2. Formulation of the Problem

We consider the diffraction of an E-polarized plane wave by a thin material strip as shown in Fig. 1, where the relative permittivity and permeability of the strip are denoted by ε_r and μ_r, respectively. Let the total electric field $\phi^t(x,z)\,[\equiv E_y(x,z)]$ be

$$\phi^t(x,z) = \phi^i(x,z) + \phi(x,z), \tag{1}$$

where $\phi^i(x,z)$ is the incident field given by

$$\phi^i(x,z) = e^{-ik(x\sin\theta_0 + z\cos\theta_0)}, \quad 0 < \theta_0 < \pi/2 \tag{2}$$

with $k\ (= \omega\sqrt{\mu_0\varepsilon_0})$ being the free-space wavenumber, and $\phi(x,z)$ is the unknown scattered field. According to the results presented in [5], the material strip is approximately replaced by a strip of zero thickness satisfying the second order impedance boundary conditions provided that the thickness of the strip is sufficiently small compared with the wavelength. On the strip surface, the total field satisfies the approximate boundary conditions as given by

$$[H_z(+0,z) + H_z(-0,z)] + 2R_m\,[E_y(+0,z) - E_y(-0,z)] = 0, \tag{3a}$$

$$\left[\frac{1}{2R_e} + \frac{1}{2\tilde{R}_m}\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right)\right][E_y(+0,z) + E_y(-0,z)] + [H_z(+0,z) - H_z(-0,z)] = 0, \tag{3b}$$

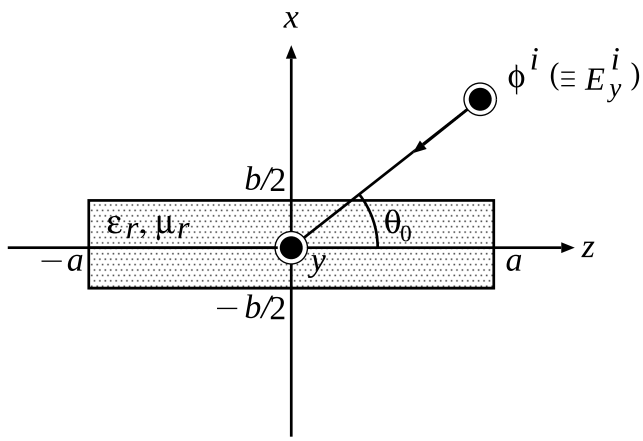

Fig. 1. Geometry of the problem.

where

$$R_e = iZ_0/[kb(\varepsilon_r - 1)], \quad R_m = iY_0/[kb(\mu_r - 1)], \quad \tilde{R}_m = i\mu_r Z_0/[kb(\mu_r - 1)] \tag{4}$$

with Z_0 and Y_0 being the intrinsic impedance and admittance of free space, respectively. In the following, we shall assume that vacuum is slightly lossy as in $k = k_1 + ik_2$ with $0 < k_2 \ll k_1$. The solution for real k is obtained by letting $k_2 \to +0$ at the end of analysis.

Let us define the Fourier transform of the scattered field $\phi(x, z)$ in (1) with respect to z as

$$\Phi(x, \alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x, z) e^{i\alpha z} dz, \quad \alpha = \text{Re } \alpha + i\text{Im } \alpha \ (\equiv \sigma + i\tau). \tag{5}$$

Then we see with the aid of the radiation condition that $\Phi(x, \alpha)$ is regular in the strip $|\tau| < k_2 \cos\theta_0$ of the complex α-plane. Introducing the Fourier integrals as

$$\Phi_\pm(x, \alpha) = \pm \frac{1}{\sqrt{2\pi}} \int_{\pm a}^{\pm\infty} \phi(x, z) e^{i\alpha(z \mp a)} dz, \tag{6a}$$

$$\Phi_1(x, \alpha) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \phi(x, z) e^{i\alpha z} dz, \tag{6b}$$

it is found that $\Phi_+(x, \alpha)$ and $\Phi_-(x, \alpha)$ are regular in the half-planes $\tau > -k_2 \cos\theta_0$ and $\tau < k_2 \cos\theta_0$, respectively, whereas $\Phi_1(x, \alpha)$ is an entire function. Using the notations as given by (6a,b), we may express $\Phi(x, \alpha)$ as

$$\Phi(x, \alpha) = e^{-i\alpha a} \Phi_-(x, \alpha) + \Phi_1(x, \alpha) + e^{i\alpha a} \Phi_+(x, \alpha). \tag{7}$$

Taking the Fourier transform of the two-dimensional Helmholtz equation and solving the resultant transformed wave equation, we derive that

$$\Phi(x, \alpha) = \left\{ -\frac{ikZ_0 \left[e^{-i\alpha a} U_-(\alpha) + e^{i\alpha a} U_{(+)}(\alpha) \right]}{2\gamma - ikZ_0 (1/R_e + \gamma^2/\tilde{R}_m k^2)} \mp \frac{e^{-i\alpha a} V_-(\alpha) + e^{i\alpha a} V_{(+)}(\alpha)}{\gamma - 2ikZ_0 R_m} \right\} e^{\mp\gamma x} \tag{8}$$

for $x \gtrless 0$, where $\gamma = \sqrt{\alpha^2 - k^2}$ with $\text{Re}\,\gamma > 0$, and

$$\begin{aligned} U_{\underset{-}{(+)}}(\alpha) = {} & \left(\frac{1}{R_e} + \frac{1}{\tilde{R}_m} \right) \Phi_\pm(0, \alpha) + \frac{1}{\tilde{R}_m k^2} \frac{d^2 \Phi_\pm(0, \alpha)}{dx^2} \\ & \mp \frac{1}{(2\pi)^{1/2} i} \left(\frac{1}{R_e} + \frac{\cos^2\theta_0}{\tilde{R}_m} \right) \frac{e^{\mp ika\cos\theta_0}}{\alpha - k\cos\theta_0}, \end{aligned} \tag{9a}$$

$$V_{\underset{-}{(+)}}(\alpha) = \frac{d\Phi_\pm(0, \alpha)}{dx} \pm \frac{k\sin\theta_0}{\sqrt{2\pi}} \frac{e^{\mp ika\cos\theta_0}}{\alpha - k\cos\theta_0}. \tag{9b}$$

Equation (8) is the scattered field representation in the Fourier transform domain. Taking into account the boundary conditions and carrying out some manipulations, we obtain from (8) that

$$-K(\alpha) J_m(\alpha) = 2[e^{-i\alpha a} V_-(\alpha) + e^{i\alpha a} V_{(+)}(\alpha)], \tag{10a}$$

$$M(\alpha) J_e(\alpha) = e^{-i\alpha a} U_-(\alpha) + e^{i\alpha a} U_{(+)}(\alpha), \tag{10b}$$

where

$$K(\alpha) = \gamma - 2ikZ_0R_m, \quad M(\alpha) = 1 - \frac{ikZ_0}{\gamma}\left[\frac{1}{2R_e} + \frac{1}{2\tilde{R}_m}\left(1 + \frac{\gamma^2}{k^2}\right)\right], \tag{11}$$

$$J_m(\alpha) = \Phi_1(+0,\alpha) - \Phi_1(-0,\alpha), \quad J_e(\alpha) = \frac{d\Phi_1(+0,\alpha)}{dx} - \frac{d\Phi_1(-0,\alpha)}{dx}. \tag{12}$$

Equations (10a,b) are the desired Wiener–Hopf equations satisfied by the unknown spectral functions and hold in the strip $|\tau| < k_2 \cos\theta_0$.

3. Exact and Asymptotic Solutions

The kernel functions $K(\alpha)$ and $M(\alpha)$ defined by (11) are factorized as

$$K(\alpha) = K_+(\alpha)K_-(\alpha) = K_+(\alpha)K_+(-\alpha), \tag{13a}$$

$$M(\alpha) = M_+(\alpha)M_-(\alpha) = M_+(\alpha)M_+(-\alpha), \tag{13b}$$

where $K_\pm(\alpha)$ and $M_\pm(\alpha)$ are split functions regular and nonzero in $\tau \gtrless \pm k_2$ (see [9] for definition). We multiply both sides of (10a) and (10b) by $e^{\pm i\alpha a}/K_\mp(\alpha)$ and $e^{\pm i\alpha a}/M_\mp(\alpha)$, respectively, and apply the decomposition procedure. This leads to

$$V_{\substack{(+)\\-}}(\alpha) = -\frac{K_\pm(\alpha)}{\pi i}\int_k^{k+i\infty} \frac{e^{2i\beta a}K_+(\beta)V_{\substack{-\\(+)}}(\mp\beta)}{\beta \pm \alpha} \frac{\sqrt{\beta^2 - k^2}}{(\beta^2 - k^2 + 4k^2Z_0^2R_m^2)} d\beta \\ \pm \frac{K_\pm(\alpha)k\sin\theta_0 e^{\mp ika\cos\theta_0}}{\sqrt{2\pi}(\alpha \pm k\cos\theta_0)K_\pm(k\cos\theta_0)}, \tag{14a}$$

$$U_{\substack{(+)\\-}}(\alpha) = -\frac{M_\pm(\alpha)}{\pi i}\int_k^{k+i\infty} \frac{e^{2i\beta a}M_+(\beta)U_{\substack{-\\(+)}}(\mp\beta)}{\beta \pm \alpha} \\ \cdot \frac{\sqrt{\beta^2 - k^2}ikZ_0[1/2R_e + (\beta^2 - k^2)/2\tilde{R}_m k^2]}{\beta^2 - k^2 + k^2Z_0^2[1/2R_e + (\beta^2 - k^2)/2\tilde{R}_m k^2]^2} d\beta \\ \mp \left(\frac{1}{R_e} + \frac{\cos^2\theta_0}{\tilde{R}_m}\right) M_\pm(\alpha) \frac{e^{\mp ika\cos\theta_0}}{\sqrt{2\pi}i(\alpha \pm k\cos\theta_0)M_\pm(k\cos\theta_0)}. \tag{14b}$$

Equations (14a,b) are the exact solution to the Wiener–Hopf equations (10a,b), but they are formal since the branch-cut integrals with the unknown integrands $V_{(+)}(\beta)$, $V_-(-\beta)$, $U_{(+)}(\beta)$, and $U_-(-\beta)$ are involved.

Applying the asymptotic method developed in [10] to the branch-cut integrals in (14a,b), we can derive high-frequency representations of (14a,b) for large $|k|a$. Omitting the details, we derive that

$$V_{\substack{(+)\\-}}(\alpha) \sim -K_\pm(\alpha)\left\{\frac{e^{i(2ka - \pi/4)}}{4\pi kZ_0^2R_m^2\sqrt{ka}} K_+(k)C_{2,1}^v\,\Gamma_1\left[3/2\,,\,\mp 2i(\alpha \pm k)a\right] \right. \\ \left. \pm \frac{k\sin\theta_0 e^{\mp ika\cos\theta_0}}{\sqrt{2\pi}K_\pm(k\cos\theta_0)(\alpha - k\cos\theta_0)}\right\}, \tag{15a}$$

$$U_{\substack{(+)\\-}}(\alpha) \sim M_\pm(\alpha)\left\{-\frac{2e^{i(2ka + \pi/4)}}{\pi\sqrt{ka}} \frac{\widetilde{R}_m R_e}{Z_0(\widetilde{R}_m + R_e)} M_+(k)C_{2,1}^u\Gamma_1\left[3/2, \mp 2i(\alpha \pm k)a\right] \right. \\ \left. \mp \frac{(1/R_e + \cos^2\theta_0/\widetilde{R}_m)e^{\mp ika\cos\theta_0}}{\sqrt{2\pi}iM_\pm(k\cos\theta_0)(\alpha - k\cos\theta_0)}\right\} \tag{15b}$$

as $ka \to \infty$ with $\theta_0 \not\approx 0, \pi$, where

$$C_{1,2}^v = \pm\frac{4}{4 - [K_+(k)A_v(k)\Gamma_1(3/2, -4ika)]^2}\left[-\frac{K_+^2(k)A_v(k)B_v(\theta_0)e^{\pm ika\cos\theta_0}\Gamma_1(3/2, -4ika)}{2k(1 \pm \cos\theta_0)K_\mp(k\cos\theta_0)} \right. \\ \left. + \frac{B_v(\theta_0)K_+(k)e^{\mp ika\cos\theta_0}}{k(1 \mp \cos\theta_0)K_\pm(k\cos\theta_0)}\right], \tag{16a}$$

$$C_{1,2}^{u} = \frac{M_+(k)[C - B_u(\theta_0)]}{1 - [M_+(k)A_u(k)\Gamma_1(3/2, -4ika)]^2} \left[\frac{M_+(k)\Gamma_1(3/2,-4ika)e^{\pm ika\cos\theta_0}}{k(1 \pm \cos\theta_0)M_\mp(k\cos\theta_0)} - \frac{e^{\mp ika\cos\theta_0}}{k(1 \mp \cos\theta_0)M_\pm(k\cos\theta_0)} \right] \tag{16b}$$

with

$$A_v(k) = \frac{e^{i(2ka-\pi/4)}}{2\pi k Z_0^2 R_m^2 \sqrt{ka}} K_+(k), \quad A_u(k) = \frac{2e^{i(2ka+\pi/4)}\tilde{R}_m R_e}{\pi\sqrt{ka}Z_0(\tilde{R}_m + R_e)} M_+(k), \tag{17}$$

$$B_v(\theta_0) = \frac{k\sin\theta_0}{\sqrt{2\pi}}, \quad B_u(\theta_0) = \frac{\sin^2\theta_0}{\sqrt{2\pi}i\tilde{R}_m}, \quad C = \left(\frac{1}{R_e} + \frac{1}{\tilde{R}_m}\right)\frac{1}{\sqrt{2\pi i}}. \tag{18}$$

In (15a,b) and (16a,b), $\Gamma_1(\cdot,\cdot)$ is the generalized gamma function [10] defined by

$$\Gamma_m(u,v) = \int_0^\infty \frac{t^{u-1}e^{-t}}{(t+v)^m} dt \tag{19}$$

for $\mathrm{Re}\, u > 0$, $|v| > 0$, $|\arg v| < \pi$, and positive integer m. Equations (15a,b) give the high-frequency asymptotic solution of the Wiener–Hopf equations (10a,b), and hold for large $|k|a$ and $\theta_0 \not\approx 0, \pi$. The scattered field is evaluated by substituting (15a,b) into (8) and taking the inverse Fourier transform with the aid of the saddle point method.

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