Dyadic Green's Functions for Multilayered Spheroidal Structures

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1 Introduction

To analyze EM radiated fields in various geometries, the *d*yadic *G*reen's *f*unction (DGF) technique provides a straightforward way. This paper represents the dyadic Green's functions in a multilayered spheroidal structure. In terms of the spheroidal vector wave functions, the DGF's are formulated in a similar form as the unbounded DGF given earlier under the spheroidal coordinates where the nonsolenoidal contribution is extracted. Multiple reflections and transmissions are considered in the construction of the scattering DGF's. Various possibilities that the source distribution and observation point are respectively located in an arbitrarily assumed region of the multilayered structure are considered in the formulation. Applications of the DGF's in spheroidal structures presented here can be found from many practical problems such as the EM waves inside and outside a stratified prolate dielectric radome utilized to protect airborne or satellite antennas from the environmental effects [1], handy phone radiation near the layered spheroid-shaped human head [2] and rainfall attenuation of microwave signals due to oblate raindrops [3].

2 Fundamental Formulation

To analyze the EM fields in spheroidal structures, we consider a prolate spheroidal geometry of multilayers as shown in Fig. 1. Here η is an angular coordinate (ranged within $-1 \leq \eta \leq 1$), ξ is a radial one (ranged within $1 \leq \xi < \infty$), ϕ is an azimuthal one (ranged within $0 \leq \phi < 2\pi$), and each spheroidal interface is assumed to have the same interfocal distance d. Oblate spheroidal problems can be analyzed by a similar procedure presented here or by the symbolic transformation, $\xi \to \pm i\xi$, and $c \to \mp ic$ where $c = \frac{1}{2}kd$ (k is the wave propagation constant). The ranges of η and ξ in the oblate spheroidal system belong to $0 \leq \eta \leq 1$ and $-\infty \leq \xi < \infty$, respectively.

Assume that the space is divided by $N-1$ spheroidal interfaces into N regions, as shown in Fig. 1. The spheroidally stratified regions are labeled respectively as $f = 1, 2, 3, \cdots, N$. The EM radiated fields, $\boldsymbol{E}_f$ and $\boldsymbol{H}_f$ in the fth (*f*ield) region ($f = 1, 2, 3, \cdots, N$) due to the electric and magnetic current distributions $\boldsymbol{J}_s$ and $\boldsymbol{M}_s$ located in the sth (*s*ource) region ($s = 1, 2, 3, \cdots, N$), can be expressed in terms of integrals containing dyadic Green's functions as follows [4]:

$$\boldsymbol{E}_f(\boldsymbol{r}) = i\omega\mu_s \iiint_V \overline{\boldsymbol{G}}_{EJ}^{(fs)}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{J}_s(\boldsymbol{r}')dV' - \iiint_V \overline{\boldsymbol{G}}_{EM}^{(fs)}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{M}_s(\boldsymbol{r}')dV', \tag{1a}$$

$$\boldsymbol{H}_f(\boldsymbol{r}) = i\omega\varepsilon_s \iiint_V \overline{\boldsymbol{G}}_{HM}^{(fs)}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{M}_s(\boldsymbol{r}')dV' + \iiint_V \overline{\boldsymbol{G}}_{HJ}^{(fs)}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{J}_s(\boldsymbol{r}')dV', \tag{1b}$$

where the prime denotes the coordinates (ξ', η', ϕ') of the current sources $\boldsymbol{J}_s$ and $\boldsymbol{M}_s$, and V identifies the volume occupied by the sources.

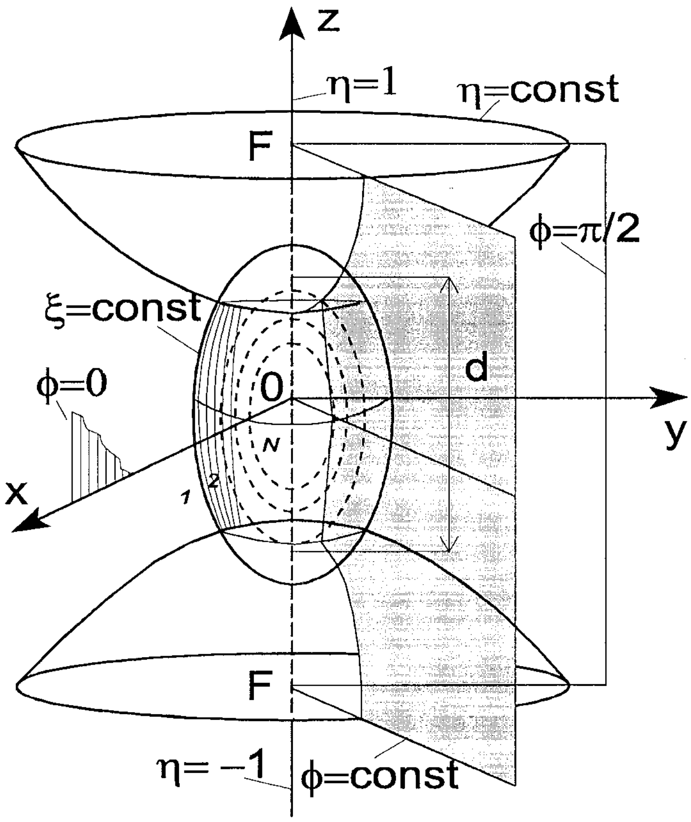

Figure 1: Geometry of a multilayered prolate spheroid under coordinates (ξ, η, ϕ)

The DGF $\overline{\boldsymbol{G}}_{HM}^{(fs)}(\boldsymbol{r}, \boldsymbol{r}')$ can be obtained from the $\overline{\boldsymbol{G}}_{EJ}^{(fs)}(\boldsymbol{r}, \boldsymbol{r}')$ by making the simple duality replacements $\boldsymbol{E} \to \boldsymbol{H}$, $\boldsymbol{H} \to -\boldsymbol{E}$, $\boldsymbol{J} \to \boldsymbol{M}$, $\boldsymbol{M} \to -\boldsymbol{J}$, $\mu \to \varepsilon$, and $\varepsilon \to \mu$. Therefore, only the DGF, $\overline{\boldsymbol{G}}_{EJ}^{(fs)}(\boldsymbol{r}, \boldsymbol{r}')$, is represented here to avoid unnecessary repetition.

3 Unbounded Dyadic Green's Functions

In terms of the defined spheroidal vector wave functions, in explicit bi-vector form, the unbounded electric dyadic Green's functions are given, for $\xi \gtrless \xi'$, as:

$$\overline{\boldsymbol{G}}_{EJ0}(\boldsymbol{r}, \boldsymbol{r}') = -\frac{\widehat{\boldsymbol{\xi}}\widehat{\boldsymbol{\xi}}}{k^2}\delta(\boldsymbol{r} - \boldsymbol{r}') + \frac{i}{2\pi}\sum_{n=m}^{\infty}\sum_{m=0}^{\infty}\frac{2-\delta_{m0}}{N_{mn}}\begin{bmatrix}\psi^{(1)}_{\substack{e\\o}mn}(c, \boldsymbol{r}') \\ \psi^{(3)}_{\substack{e\\o}mn}(c, \boldsymbol{r}')\end{bmatrix} \cdot \left\{\begin{bmatrix}\boldsymbol{N}^{x(3)}_{\substack{e\\o}mn}(c, \boldsymbol{r}) \\ \boldsymbol{N}^{x(1)}_{\substack{e\\o}mn}(c, \boldsymbol{r})\end{bmatrix}\widehat{\boldsymbol{x}} + \begin{bmatrix}\boldsymbol{N}^{y(3)}_{\substack{e\\o}mn}(c, \boldsymbol{r}) \\ \boldsymbol{N}^{y(1)}_{\substack{e\\o}mn}(c, \boldsymbol{r})\end{bmatrix}\widehat{\boldsymbol{y}} + \begin{bmatrix}\boldsymbol{N}^{z(3)}_{\substack{e\\o}mn}(c, \boldsymbol{r}) \\ \boldsymbol{N}^{z(1)}_{\substack{e\\o}mn}(c, \boldsymbol{r})\end{bmatrix}\widehat{\boldsymbol{z}}\right\}, \tag{2}$$

where $\widehat{\boldsymbol{\xi}}$ denotes the spheroidal radial unit vector, $\delta(\boldsymbol{r} - \boldsymbol{r}')$ is the three-dimensional Dirac delta function, and the prime denotes the coordinates (ξ', η', ϕ'). The first term of Eq. (2) stands for the nonsolenoidal contribution and can be obtained by using the same method given by Tai [4]. The spheroidal vector wave functions $\boldsymbol{M}^{a(i)}_{\substack{e\\o}mn}(c, \boldsymbol{r})$ and $\boldsymbol{N}^{a(i)}_{\substack{e\\o}mn}(c, \boldsymbol{r})$ $(\widehat{\boldsymbol{a}} = \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}})$ for the construction of Green's dyadics are defined in terms of the scalar eigenfunctions as follows:

$$\boldsymbol{M}^{a(i)}_{\substack{e\\o}mn}(c, \boldsymbol{r}) = \boldsymbol{\nabla} \times \left[\psi^{(i)}_{\substack{e\\o}mn}(c, \boldsymbol{r})\widehat{\boldsymbol{a}}\right], \quad \widehat{\boldsymbol{a}} = \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}, \tag{3a}$$

$$\boldsymbol{N}^{a(i)}_{\substack{e\\o}mn}(c,\boldsymbol{r}) = \frac{1}{k}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\left[\psi^{(i)}_{\substack{e\\o}mn}(c,\boldsymbol{r})\widehat{\boldsymbol{a}}\right]. \tag{3b}$$

The explicit forms of the spheroidal vector wave functions under the alternative spheroidal coordinates system are given by Flammar in [5].

4 Scattering Green's Dyadics

Using the principle of scattering superposition, the dyadic Green's function can be considered as the sum of the unbounded Green's dyadic and a scattering Green's dyadic to be determined. The Green's dyadic is therefore given by [4]:

$$\overline{\boldsymbol{G}}^{(fs)}_{EJ}(\boldsymbol{r},\boldsymbol{r}') = \overline{\boldsymbol{G}}_{EJ0}(\boldsymbol{r},\boldsymbol{r}')\delta_{fs} + \overline{\boldsymbol{G}}^{(fs)}_{EJs}(\boldsymbol{r},\boldsymbol{r}'), \tag{4}$$

where the scattering DGF $\overline{\boldsymbol{G}}^{(fs)}_{EJs}(\boldsymbol{r},\boldsymbol{r}')$ describes an additional contribution of the multiple reflection and transmission waves in the presence of the boundary produced by the dielectric media while the unbounded dyadic Green's function, $\overline{\boldsymbol{G}}_{EJ0}(\boldsymbol{r},\boldsymbol{r}')$, given by (2) represents the contribution of the direct waves from radiation sources in an unbounded medium. The superscript (fs) denotes the layers where the *f*ield point and the *s*ource point are located, respectively, and the subscript s identifies the *s*cattering dyadic Green's functions.

When the antenna is located in the sth region, the scattering dyadic Green's function in the fth regions must be of the form similar to that of the unbounded Green's dyadic. To satisfy the boundary conditions, however, the additional spheroidal vector wave functions $\boldsymbol{M}^{a(i)}_{\substack{o\\e}mn}(c,\xi)$ should be included to account for the effects of multiple transmissions and reflections. For the ease of determination of the scattering coefficients, the sets of vector wave functions, $\boldsymbol{M}^{\pm(1)}_{\substack{e\\o}m\pm1,n}(c,\xi)$ and $\boldsymbol{N}^{\pm(i)}_{\substack{e\\o}m\pm1,n}(c,\xi)$, are used in the construction of the scattering DGF's. $\boldsymbol{M}^{\pm(i)}_{\substack{e\\o}m\pm1,n}(c,\xi)$ and $\boldsymbol{N}^{\pm(i)}_{\substack{e\\o}m\pm1,n}(c,\xi)$ are defined as follows:

$$\boldsymbol{X}^{+(i)}_{\substack{e\\o}m+1,n}(c,\xi) = \frac{1}{2}\left[\boldsymbol{X}^{x(i)}_{\substack{e\\o}mn}(c,\xi) \mp \boldsymbol{X}^{y(i)}_{\substack{o\\e}mn}(c,\xi)\right], \tag{5a}$$

$$\boldsymbol{X}^{-(i)}_{\substack{e\\o}m-1,n}(c,\xi) = \frac{1}{2}\left[\boldsymbol{X}^{x(i)}_{\substack{e\\o}mn}(c,\xi) \pm \boldsymbol{X}^{y(i)}_{\substack{o\\e}mn}(c,\xi)\right], \tag{5b}$$

where $\boldsymbol{X}$ denotes either $\boldsymbol{M}$ or $\boldsymbol{N}$.

For a single-layered spheroidal geometry, the dyadic Green's functions have been given by Li *et al.* [6] . Therefore, the scattering dyadic Green's functions in each region of a multilayered spheroidal structure can be formulated in a similar fashion.

$$\begin{aligned}
\overline{\boldsymbol{G}}^{(fs)}_{EJs}(\boldsymbol{r},\boldsymbol{r}') = \frac{i}{2\pi}\sum_{n=m}^{\infty}\sum_{m=0}^{\infty}\Bigg\{&(1-\delta_{fN})\cdot\Bigg[\Bigg(\mathcal{A}^{+xM}_{f\substack{o\\e}mn}\boldsymbol{M}^{+(3)}_{\substack{o\\e}m+1,n}(c_f,\xi) + \mathcal{A}^{+xN}_{f\substack{e\\o}mn}\boldsymbol{N}^{+(3)}_{\substack{e\\o}m+1,n}(c_f,\xi)\\
&+ \mathcal{A}^{-xM}_{f\substack{o\\e}mn}\boldsymbol{M}^{-(3)}_{\substack{o\\e}m-1,n}(c_f,\xi) + \mathcal{A}^{-xN}_{f\substack{e\\o}mn}\boldsymbol{N}^{-(3)}_{\substack{e\\o}m-1,n}(c_f,\xi)\Bigg)\widehat{\boldsymbol{x}}\\
&+\Bigg(\mathcal{A}^{+yM}_{f\substack{e\\o}mn}\boldsymbol{M}^{+(3)}_{\substack{e\\o}m+1,n}(c_f,\xi) + \mathcal{A}^{+yN}_{f\substack{o\\e}mn}\boldsymbol{N}^{+(3)}_{\substack{o\\e}m+1,n}(c_f,\xi)\\
&+ \mathcal{A}^{-yM}_{f\substack{e\\o}mn}\boldsymbol{M}^{-(3)}_{\substack{e\\o}m-1,n}(c_f,\xi) + \mathcal{A}^{-yN}_{f\substack{o\\e}mn}\boldsymbol{N}^{-(3)}_{\substack{o\\e}m-1,n}(c_f,\xi)\Bigg)\widehat{\boldsymbol{y}}\\
&+\Bigg(\mathcal{A}^{zM}_{f\substack{o\\e}mn}\boldsymbol{M}^{z(3)}_{\substack{o\\e}mn}(c_f,\xi) + \mathcal{A}^{zN}_{f\substack{e\\o}mn}\boldsymbol{N}^{z(3)}_{\substack{e\\o}mn}(c_f,\xi)\Bigg)\widehat{\boldsymbol{z}}\Bigg]\\
&+(1-\delta_{fN})\cdot\Bigg[\Bigg(\mathcal{B}^{+xM}_{f\substack{o\\e}mn}\boldsymbol{M}^{+(1)}_{\substack{o\\e}m+1,n}(c_f,\xi) + \mathcal{B}^{+xN}_{f\substack{e\\o}mn}\boldsymbol{N}^{+(1)}_{\substack{e\\o}m+1,n}(c_f,\xi)
\end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{B}^{-xM}_{f\substack{o\\e}mn} \boldsymbol{M}^{-(1)}_{\substack{o\\e}m-1,n}(c_f, \xi) + \mathcal{B}^{-xN}_{f\substack{e\\o}mn} \boldsymbol{N}^{-(1)}_{\substack{e\\o}m-1,n}(c_f, \xi) \Big) \,\widehat{\boldsymbol{x}} \\
&+ \Big(\mathcal{B}^{+yM}_{f\substack{e\\o}mn} \boldsymbol{M}^{+(1)}_{\substack{e\\o}m+1,n}(c_f, \xi) + \mathcal{B}^{+yN}_{f\substack{o\\e}mn} \boldsymbol{N}^{+(1)}_{\substack{o\\e}m+1,n}(c_f, \xi) \\
&+ \mathcal{B}^{-yM}_{f\substack{e\\o}mn} \boldsymbol{M}^{-(1)}_{\substack{e\\o}m-1,n}(c_f, \xi) + \mathcal{B}^{-yN}_{f\substack{o\\e}mn} \boldsymbol{N}^{-(1)}_{\substack{o\\e}m-1,n}(c_f, \xi) \Big) \,\widehat{\boldsymbol{y}} \\
&+ \Big(\mathcal{B}^{zM}_{f\substack{o\\e}mn} \boldsymbol{M}^{z(1)}_{\substack{o\\e}mn}(c_f, \xi) + \mathcal{B}^{zN}_{f\substack{e\\o}mn} \boldsymbol{N}^{z(1)}_{\substack{e\\o}mn}(c_f, \xi) \Big) \,\widehat{\boldsymbol{z}} \Big]\Big] \Big\}. \qquad (6)
\end{aligned}
$$

Here δ_{fN} and δ_{f1} are Kronecker delta functions. $c_s = \frac{1}{2}k_s d$ and $c_f = \frac{1}{2}k_f d$, where k_s and k_f are respectively the wave propagation constants in which the source and field points are located. $\mathcal{A}^{(\pm x, \pm y, z)(M,N)}_{f\substack{e\\o}mn}$ and $\mathcal{B}^{(\pm x, \pm y, z)(M,N)}_{f\substack{e\\o}mn}$ are unknown scattering coefficients to be determined from the boundary conditions. Because of the nonorthogonality of the spheroidal wave functions, these unknown coefficients are coupled to each other. By using the method of functional expansion [6], the coupled unknowns can be determined from the matrix equation system exclusively. Sufficient computational accuracy can be achieved by properly choosing the truncation number of the matrix. Elements of the matrix equation system are provided in detail in [6].

5 Discussion and Conclusion

In this paper, the dyadic Green's functions in multilayered spheroidal structures are formulated in terms of appropriate vector wave eigenfunctions and coordinate vectors $\widehat{\boldsymbol{x}}$, $\widehat{\boldsymbol{y}}$, and $\widehat{\boldsymbol{z}}$. The representation and formulation of DGF's in the spheroidal coordinates are made in a different eigenfunction expansion, as compared with those conventional formulations in planar, cylindrical, and spherical coordinates. The nonsolenoidal term of the electric dyadic Green's function is extracted by following the same procedure given by Tai [4]. The unknown coefficients of the scattering dyadic Green's functions, even coupled to each other, can be determined from the matrix equation system using the functional expansion technique.

References

[1] J.D. Walton Jr., Ed., *Radome Engineering Handbook: Design and Principles*, New York: Marcel Dekker, Inc., 1970.

[2] X.K. Kang, L.W. Li, M.S. Leong, and P.S. Kooi, "A spheroidal vector wave function analysis of field and SAR distributions in a dielectric prolate spheroidal human head model", *J. Electromagn. Waves Applic.*, vol. 13, pp. 651–653, 1999.

[3] J.A. Morrison and M.J. Cross, "Scattering of a plane electromagnetic wave by axisymmetric raindrops", *Bell Syst. Tech. J.*, vol. 53, pp. 955–1019, 1974.

[4] C.T. Tai, *Dyadic Green's Functions in Electromagnetic Theory*, Piscataway, New Jersey: IEEE Press, The 2nd edition, 1994.

[5] C. Flammer, *Spheroidal wave functions*, California: Stanford Univ. Press, 1957.

[6] L.W. Li, M.S. Leong, P.S. Kooi, and T.S. Yeo, "Spheroidal vector wave eigenfunction expansion of dyadic Green's functions for a dielectric spheroid", *IEEE Trans. Antennas Propagt.*, in press, 2000.