Electromagnetic Radiation Field from a Flanged Rectangular Waveguide

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1. Introduction

The analysis of radiation from a waveguide is essentially the same as the diffraction of a wave by a rectangular hole in an infinite plate and it may be rigorously treated by applying the method of the Kobayashi potential (KP).[1]−[3] For the two-dimensional problem, one of the authors has derived an exact expression of the field radiated from a flanged parallel-plate waveguide via the method of the KP and some numerical results were presented.[4]

In this paper, the radiation of an electromagnetic wave from a flanged rectangular waveguide is exactly formulated by using the KP method when the waveguide is excited by TE- and TM-mode waves. The process of analysis is similar to [4]. The fields in the waveguide and half-space are expanded in terms of the normal modes of the waveguide and the Weber-Schafheitlin (WS) discontinuous integrals, respectively. Imposing the continuity of the tangential components of the fields on the aperture, we obtain a matrix equation to determine the expansion coefficients of the diffracted wave. The matrix elements are given by double infinite integrals and double infinite series which contain Bessel functions, and an effective method for computing the double infinite series is proposed. Using the derived formula for the far diffracted field, the radiation patterns are computed for various kinds of parameters.

2. Statement of the problem

Fig. 1 shows an open-ended flanged rectangular waveguide and the related coordinate systems. We study the field $\mathbf{E}^d$ radiated from the aperture to the half-space $z > 0$, when an electromagnetic wave $\mathbf{E}^i$ in the waveguide are propagating in the positive z direction. The flange and waveguide are assumed to be perfectly conducting and the permittivity and permeability of the half-space (region I) are ϵ_1 and μ_1, respectively. The inside of the waveguide (region II) is filled with a homogeneous isotropic medium with parameters ϵ_2, μ_2, and the dimension of the guide is $2a \times 2b$. The aperture and flange are located in the $z = 0$ plane. Because of the discontinuity of the geometry at $z = 0$, the reflected wave of the incident wave and the higher-order mode waves are excited at the open end. In this analysis, the harmonic time dependence $\exp(j\omega t)$ is assumed and omitted throughout.

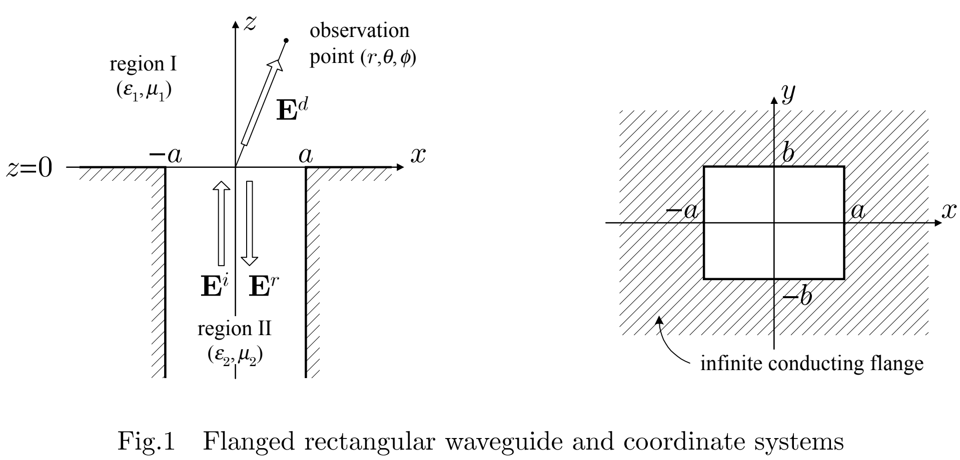

Fig.1 Flanged rectangular waveguide and coordinate systems

A. Field in the waveguide

For simplicity, we separate the incident wave $\mathbf{E}^i$ in the waveguide into TE- and TM-mode waves. The reflected wave $\mathbf{E}^r$ including the higher-order mode waves is represented as linear combination of the

TE and TM modes. In this analysis, we use the z component of vetor potentials for the incident and reflected waves. The electric vector potential F_z is used for TE mode and the magnetic vector potential A_z is for TM mode. These potentials are expanded by eigenmode functions of the waveguide as follows.

$$\text{TE mode:} \quad \begin{pmatrix} F_z^i \\ F_z^r \end{pmatrix} = a\epsilon_2 \sum_{\substack{m=0 \\ (m,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \begin{pmatrix} A_{mn}^{(E)} \exp(-jh_{mn}z_a) \\ B_{mn}^{(E)} \exp(jh_{mn}z_a) \end{pmatrix} \cos\left[\frac{m\pi}{2}(\xi+1)\right] \cos\left[\frac{n\pi}{2}(\eta+1)\right] \tag{1}$$

$$\text{TM mode:} \quad \begin{pmatrix} A_z^i \\ A_z^r \end{pmatrix} = \frac{\kappa_2^2}{\omega} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{pmatrix} A_{mn}^{(M)} \exp(-jh_{mn}z_a) \\ B_{mn}^{(M)} \exp(jh_{mn}z_a) \end{pmatrix} \sin\left[\frac{m\pi}{2}(\xi+1)\right] \sin\left[\frac{n\pi}{2}(\eta+1)\right] \tag{2}$$

where $\xi = x/a$, $\eta = y/b$, $z_a = z/a$ are the normalized variables, and the other symbols are defined by

$$h_{mn} = \sqrt{\kappa_2^2 - (m\pi/2)^2 - p^2(n\pi/2)^2}, \quad p = a/b\,(=1/q)\,, \quad \kappa_2 = k_2 a, \quad k_2 = \omega\sqrt{\epsilon_2\mu_2}. \tag{3}$$

B. Expression of the diffracted field

We use the x and y components of the electric vector potential $\mathbf{F}$ for the diffracted waves in the half-space, and these are expressed by using the Fourier transform of the Helmholtz equation. The expressions include some unknown functions which are determined by the following boundary conditions.

$$E_x^d = 0, \quad E_y^d = 0, \quad (x,y) \in D^c, \quad z = +0 \tag{4a}$$

$$E_x^d = E_x^i + E_x^r, \quad E_y^d = E_y^i + E_y^r, \quad (x,y) \in D, \quad z = 0 \tag{4b}$$

$$H_x^d = H_x^i + H_x^r, \quad H_y^d = H_y^i + H_y^r, \quad (x,y) \in D, \quad z = 0. \tag{4c}$$

Here $D = \{(x,y) \mid |x| < a, \ |y| < b\} \subset \mathbf{R}^2$ represents the domain of the aperture, and D^c is the complement of D, that is, the region on the flange. Eqs. (4b) and (4c) express the continuity of the tangential components of the electric and magnetic fields on the aperture, respectively. Eq. (4a) is the condition that the tangential electric field must vanish on the flange, and it is automatically satisfied by representing the field by the WS integrals. Therefore, the resultant expressions of F_x^d and F_y^d are given as follows.

$$\begin{aligned} F_x^d = a\epsilon_1 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \int_0^{\infty}\int_0^{\infty} \frac{1}{\zeta(\alpha,\beta)} \Big\{ & \Lambda_{2m}^{\sigma}(\alpha)\cos\alpha\xi \left[A_{mn}^{(x)}\Lambda_{2n}^{\tau}(\beta)\cos\beta\eta + B_{mn}^{(x)}\Lambda_{2n+1}^{\tau}(\beta)\sin\beta\eta\right] \\ & + \Lambda_{2m+1}^{\sigma}(\alpha)\sin\alpha\xi \left[C_{mn}^{(x)}\Lambda_{2n}^{\tau}(\beta)\cos\beta\eta + D_{mn}^{(x)}\Lambda_{2n+1}^{\tau}(\beta)\sin\beta\eta\right]\Big\} \exp\left[-\zeta(\alpha,\beta)\, z_a\,\right] d\alpha d\beta \end{aligned} \tag{5a}$$

$$\begin{aligned} F_y^d = a\epsilon_1 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \int_0^{\infty}\int_0^{\infty} \frac{1}{\zeta(\alpha,\beta)} \Big\{ & \Lambda_{2m}^{\tau}(\alpha)\cos\alpha\xi \left[A_{mn}^{(y)}\Lambda_{2n}^{\sigma}(\beta)\cos\beta\eta + B_{mn}^{(y)}\Lambda_{2n+1}^{\sigma}(\beta)\sin\beta\eta\right] \\ & + \Lambda_{2m+1}^{\tau}(\alpha)\sin\alpha\xi \left[C_{mn}^{(y)}\Lambda_{2n}^{\sigma}(\beta)\cos\beta\eta + D_{mn}^{(y)}\Lambda_{2n+1}^{\sigma}(\beta)\sin\beta\eta\right]\Big\} \exp\left[-\zeta(\alpha,\beta)\, z_a\,\right] d\alpha d\beta \end{aligned} \tag{5b}$$

$$\Lambda_\ell^{\nu}(x) = J_{\ell+\nu}(x)/x^{\nu}, \quad \zeta(\alpha,\beta) = \sqrt{\alpha^2 + p^2\beta^2 - \kappa_1^2}, \quad \kappa_1 = k_1 a, \quad k_1 = \omega\sqrt{\epsilon_1\mu_1}. \tag{5c}$$

The parameters σ and τ, which affect the convergence of the solution,[5] have some arbitrariness to choose. If we select $\sigma = 1$ and $\tau = 0$, Eq. (5) coincides with that of the thin rectangular aperture.[3]

C. Matrix equation

By enforcing the remaining conditions (4b) and (4c), we can get a determinantal equation for the expansion coefficients $A_{mn}^{(x)}$ through $D_{mn}^{(x)}$ and $A_{mn}^{(y)}$ through $D_{mn}^{(y)}$ as a matrix equation.

(1) TE-mode wave incidence

Imposing the condition (4b) to the tangential electric fields on the aperture and separating the variable ξ and η by using the orthogonality of trigonometric functions, one relation of the expansion coefficients is obtained. Next, imposing the condition (4c) of the tangential magnetic field and applying the orthogonality of the Jacobi's polynomials,[3] we obtain another relation for the coefficients. From these relations, the matrix equation for the expansion coefficients is derived as below.

$$\begin{aligned} &\begin{bmatrix} K_{Amnst}^{\sigma\tau}(u,v) + R_\mu KS_{Amnst}^{\sigma\tau}(u,v) & (-1)^{u+v}p\{G_{Amnst}^{\sigma\tau}(u,v) + R_\mu GS_{Amnst}^{\sigma\tau}(u,v)\} \\ (-1)^{u+v}q\{G_{Bmnst}^{\sigma\tau}(\bar{u},\bar{v}) + R_\mu GS_{Bmnst}^{\sigma\tau}(\bar{u},\bar{v})\} & K_{Bmnst}^{\sigma\tau}(\bar{u},\bar{v}) + R_\mu KS_{Bmnst}^{\sigma\tau}(\bar{u},\bar{v}) \end{bmatrix} \\ &\times \begin{bmatrix} X_{mn}^{(x)}(u,v) \\ X_{mn}^{(y)}(\bar{u},\bar{v}) \end{bmatrix} = 2jR_\mu(-1)^{u+v} \begin{bmatrix} P_{st}^{(x)}(u,v) \\ P_{st}^{(y)}(\bar{u},\bar{v}) \end{bmatrix} \qquad s = 0,1,2,\cdots, \quad t = 0,1,2,\cdots. \end{aligned} \tag{6}$$

where $\bar{u} = 1-u$, $\bar{v} = 1-v$, and $(u,v) \in \{(0,0),(0,1),(1,0),(1,1)\}$. The other symbols are defined by

$$X^{(i)}_{mn}(0,0) = A^{(i)}_{mn}, \quad X^{(i)}_{mn}(0,1) = B^{(i)}_{mn}, \quad X^{(i)}_{mn}(1,0) = C^{(i)}_{mn}, \quad X^{(i)}_{mn}(1,1) = D^{(i)}_{mn}, \quad (i = x \text{ or } y) \tag{7a}$$

$$P^{(x)}_{st}(u,v) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}h_{2m+1+u,2n+v}A^{(E)}_{2m+1+u,2n+v}J_{2s+1+u}\left(\tfrac{2m+1+u}{2}\pi\right)\Lambda_{2t+1+v}\left(\tfrac{2n+v}{2}\pi\right) \tag{7b}$$

$$P^{(y)}_{st}(u,v) = q\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}h_{2m+u,2n+1+v}A^{(E)}_{2m+u,2n+1+v}\Lambda_{2s+1+u}\left(\tfrac{2m+u}{2}\pi\right)J_{2t+1+v}\left(\tfrac{2n+1+v}{2}\pi\right) \tag{7c}$$

$$\Lambda_\ell(x) = J_\ell(x)/x, \quad R_\mu = \mu_1/\mu_2. \tag{7d}$$

The matrix elements consist of double infinite integrals and double infinite series, which are given by

$$K^{\sigma\tau}_{Amnst}(u,v) = \int_0^\infty\int_0^\infty \frac{\kappa_1^2-\alpha^2}{\zeta(\alpha,\beta)}\Lambda^{\sigma}_{2m+u}(\alpha)\Lambda_{2s+1+u}(\alpha)\Lambda^{\tau}_{2n+v}(\beta)\Lambda_{2t+1+v}(\beta)d\alpha d\beta \tag{8a}$$

$$K^{\sigma\tau}_{Bmnst}(u,v) = \int_0^\infty\int_0^\infty \frac{q^2\kappa_1^2-\beta^2}{\zeta(\alpha,\beta)}\Lambda^{\tau}_{2m+u}(\alpha)\Lambda_{2s+1+u}(\alpha)\Lambda^{\sigma}_{2n+v}(\beta)\Lambda_{2t+1+v}(\beta)d\alpha d\beta \tag{8b}$$

$$G^{\sigma\tau}_{Amnst}(u,v) = \int_0^\infty\int_0^\infty \frac{1}{\zeta(\alpha,\beta)}\Lambda^{\tau}_{2m+1-u}(\alpha)J_{2s+1+u}(\alpha)\Lambda^{\sigma}_{2n+1-v}(\beta)J_{2t+1+v}(\beta)d\alpha d\beta \tag{8c}$$

$$G^{\sigma\tau}_{Bmnst}(u,v) = \int_0^\infty\int_0^\infty \frac{1}{\zeta(\alpha,\beta)}\Lambda^{\sigma}_{2m+1-u}(\alpha)J_{2s+1+u}(\alpha)\Lambda^{\tau}_{2n+1-v}(\beta)J_{2t+1+v}(\beta)d\alpha d\beta \tag{8d}$$

$$KS^{\sigma\tau}_{Amnst}(u,v) = \pi^2\sum_{m'=0}^{\infty}\sum_{n'=0}^{\infty}\frac{\kappa_2^2-\left(\frac{2m'+1+u}{2}\pi\right)^2}{(1+\delta_{0,2n'+v})\gamma_{2m'+1+u,2n'+v}}$$
$$\times\Lambda^{\sigma}_{2m+u}\left(\tfrac{2m'+1+u}{2}\pi\right)\Lambda_{2s+1+u}\left(\tfrac{2m'+1+u}{2}\pi\right)\Lambda^{\tau}_{2n+v}\left(\tfrac{2n'+v}{2}\pi\right)\Lambda_{2t+1+v}\left(\tfrac{2n'+v}{2}\pi\right) \tag{9a}$$

$$KS^{\sigma\tau}_{Bmnst}(u,v) = \pi^2\sum_{m'=0}^{\infty}\sum_{n'=0}^{\infty}\frac{q^2\kappa_2^2-\left(\frac{2n'+1+v}{2}\pi\right)^2}{(1+\delta_{0,2m'+u})\gamma_{2m'+u,2n'+1+v}}$$
$$\times\Lambda^{\tau}_{2m+u}\left(\tfrac{2m'+u}{2}\pi\right)\Lambda_{2s+1+u}\left(\tfrac{2m'+u}{2}\pi\right)\Lambda^{\sigma}_{2n+v}\left(\tfrac{2n'+1+v}{2}\pi\right)\Lambda_{2t+1+v}\left(\tfrac{2n'+1+v}{2}\pi\right) \tag{9b}$$

$$GS^{\sigma\tau}_{Amnst}(u,v) = \pi^2\sum_{m'=0}^{\infty}\sum_{n'=0}^{\infty}\frac{1}{\gamma_{2m'+1+u,2n'+v}}$$
$$\times\Lambda^{\tau}_{2m+1-u}\left(\tfrac{2m'+1+u}{2}\pi\right)J_{2s+1+u}\left(\tfrac{2m'+1+u}{2}\pi\right)\Lambda^{\sigma}_{2n+1-v}\left(\tfrac{2n'+v}{2}\pi\right)J_{2t+1+v}\left(\tfrac{2n'+v}{2}\pi\right) \tag{9c}$$

$$GS^{\sigma\tau}_{Bmnst}(u,v) = \pi^2\sum_{m'=0}^{\infty}\sum_{n'=0}^{\infty}\frac{1}{\gamma_{2m'+u,2n'+1+v}}$$
$$\times\Lambda^{\sigma}_{2m+1-u}\left(\tfrac{2m'+u}{2}\pi\right)J_{2s+1+u}\left(\tfrac{2m'+u}{2}\pi\right)\Lambda^{\tau}_{2n+1-v}\left(\tfrac{2n'+1+v}{2}\pi\right)J_{2t+1+v}\left(\tfrac{2n'+1+v}{2}\pi\right) \tag{9d}$$

where $\delta_{\ell\ell'}$ is the Kronecker delta and $\gamma_{mn} = \sqrt{(m\pi/2)^2 + p^2(n\pi/2)^2 - \kappa_2^2} = jh_{mn}$.

(2) TM-mode wave incidence

From the same process as TE mode, Eq.(6) is again obtained except for the expressions of P_{st} below.

$$P^{(x)}_{st}(u,v) = p\kappa_2^2\sum_{m=0}^{\infty}\sum_{n=1-v}^{\infty}(-1)^{m+n}A^{(M)}_{2m+1+u,2n+v}\Lambda_{2s+1+u}\left(\tfrac{2m+1+u}{2}\pi\right)J_{2t+1+v}\left(\tfrac{2n+v}{2}\pi\right) \tag{10a}$$

$$P^{(y)}_{st}(u,v) = -q^2\kappa_2^2\sum_{m=1-u}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}A^{(M)}_{2m+u,2n+1+v}J_{2s+1+u}\left(\tfrac{2m+u}{2}\pi\right)\Lambda_{2t+1+v}\left(\tfrac{2n+1+v}{2}\pi\right). \tag{10b}$$

D. Expression of the far field

The far-field expression can be obtained from Eq. (5) by applying the stationary phase method of integration,[3] and the resulting expression is given by

$$F^d_x = \frac{\pi q a^2\epsilon_1}{2}\frac{\exp(-jk_1 r)}{r}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\Lambda^{\sigma}_{2m}(\kappa_a)\left[A^{(x)}_{mn}\Lambda^{\tau}_{2n}(\kappa_b) + jB^{(x)}_{mn}\Lambda^{\tau}_{2n+1}(\kappa_b)\right]$$

$$+\ \Lambda^{\sigma}_{2m+1}(\kappa_a)\left[jC^{(x)}_{mn}\Lambda^{\tau}_{2n}(\kappa_b) - D^{(x)}_{mn}\Lambda^{\tau}_{2n+1}(\kappa_b)\right]\Big\} \tag{11a}$$

$$F^d_y = \frac{\pi q a^2 \epsilon_1}{2}\frac{\exp(-jk_1 r)}{r}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\Lambda^{\tau}_{2m}(\kappa_a)\left[A^{(y)}_{mn}\Lambda^{\sigma}_{2n}(\kappa_b) + jB^{(y)}_{mn}\Lambda^{\sigma}_{2n+1}(\kappa_b)\right]$$

$$+\ \Lambda^{\tau}_{2m+1}(\kappa_a)\left[jC^{(y)}_{mn}\Lambda^{\sigma}_{2n}(\kappa_b) - D^{(y)}_{mn}\Lambda^{\sigma}_{2n+1}(\kappa_b)\right]\Big\} \tag{11b}$$

where $\kappa_a = \kappa_1 \sin\theta\cos\phi$, $\kappa_b = q\kappa_1\sin\theta\sin\phi$. The far electric field is computed by the following relations.

$$E_\theta \simeq j(k_1/\epsilon_1)(F_x\sin\phi - F_y\cos\phi), \qquad E_\phi \simeq j(k_1/\epsilon_1)\cos\theta(F_x\cos\phi + F_y\sin\phi). \tag{12}$$

3. Computation of the matrix elements

To obtain the diffracted patterns by means of Eqs. (11) and (12), we must determine the expansion coefficients by solving the matrix equation of (6) which include double infinite integrals and double infinite series. The double infinite integral is calculated with the method of [6], and similar manner may also be applied to the computation of the double infinite series. Therefore, the asymptotic expression of the summand is used. In the process of the computation of the double infinite series for $\sigma = 1$ and $\tau = 0$, the following single and double infinite series about the algebraic functions are obtained, which are also appeared in the diffraction problem of a scalar wave by a thick rectangular aperture.[7]

$$S_\ell(x_{N+1}, \pm b^2) = \sum_{n=N+1}^{\infty}\frac{1}{x_n^\ell\sqrt{x_n^2 \pm b^2}},\ \ SW_{k\ell}(\alpha_{M+1}, \beta_{N+1}) = \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty}\frac{1}{\alpha_m^k\beta_n^\ell\sqrt{\alpha_m^2 + p^2\beta_n^2}} \tag{13a}$$

$$x_n = (n+u/2)\pi, \quad \alpha_m = (m+u/2)\pi, \quad \beta_n = (n+v/2)\pi \qquad (u, v = 0 \text{ or } 1). \tag{13b}$$

S_ℓ is calculated with the desired accuracy by using the Taylor expansion of the summand for $b < x_{N+1}$, but its expression converges fairly slowly when b approaches x_{N+1}. For the large b value, we use an integral formula of the modified Bessel function $K_0(x)$ and the trigonometric function, and S_ℓ is transformed into the sum of an infinite integral converging rapidly and some finite series. $SW_{k\ell}$ is also calculated using two formulas about the modified Bessel function. Thus, we can estimate the values of the double infinite series containing the Bessel functions, and the radiation pattern is readily obtained from Eqs. (11) and (12). For the various kinds of incident waves and aperture sizes the far-field patterns are computed, although the plots of the results are omitted here for saving space.

4. Conclusion

We derived the exact solution of the electromagnetic field radiated from a flanged rectangular waveguide by using the method of the Kobayashi potential for TE- and TM-wave incidences. The problem was reduced to the matrix equation for the expansion coefficients of the diffracted field, and the matrix elements were given by double infinite integrals and double infinite series. By using the effective method for computing these integrals and series, the radiation patterns were obtained for the various kinds of parameters. Since our results can be readily extended to the problem of a rectangular waveguide array as in the two-dimensional problem,[8] the rigolous treatment of mutual coupling between the elements will be made in the near future.

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