Mathematical Foundation of Modified Edge Representation

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Abstract

Modified Edge Representation (MER) empirically proposed by one of the authors has remarkable accuracy for the surface to line integral reduction of radiation integrals in scattering problems for near source[1]. This paper investigates the mathematical principle of MER. Some numerical results are given to show the accuracy of MER in the scattering from the rectangular.

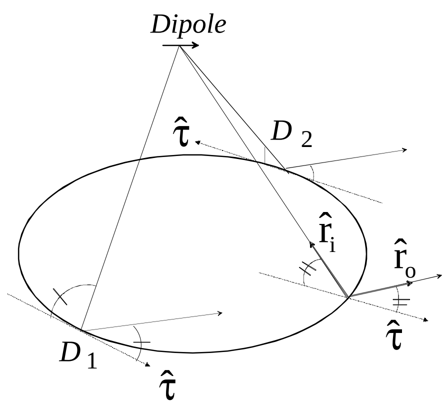
Ⅰ. INTRODUCTION

Physical optics(PO) is one of the high frequency techniques, which has been widely applied to the pattern analysis of reflector antennas. Since this technique involves the surface integral, it has been pursued the reduction of the integral to the line one for saving the numerical computation. As for this reduction problem there are two kinds of theories, namely the exact reduction theory using Stokes's theorem [2][3] and the asymptotic reduction theory. But the former has major drawback that it can't be applied to scatterers with curved surface because the image theory is used to derive the reduction. On the contrary, the asymptotic reduction theory would have the potential capable of being applied to curved scatterers and to GO approximate incidence. But it needs at least five wave lengths for the distance from the scatterer to keep the good accuracy. In this paper we will show that MER has directly derived by using Stokes's theorem without the use of the image theory, and that MER for the far field is the excellent reduction theory.

II. MODIFIED EDGE REPRESENTATION

A new concept named modified edge representation (MER) was empirically introduced by one of the authors[1] for the surface-to-edge integral reduction of PO currents. Firstly, the edge of the scatterer is replaced with modified one (unit vector $\hat{\tau}$) satisfying the diffraction law at each point as shown in Fig.1. This requirement is depicted in Fig.1 and is simply expressed as

$$(\hat{r}_o + \hat{r}_i) \cdot \hat{\tau} = 0 \quad \text{for general points} \tag{1}$$



$$\hat{\tau} = \hat{t} \quad \text{for diffraction points} \tag{2}$$

Fig.1 Direction of $\hat{\tau}$ at several edge points in MER.

The direction of vector $\hat{\tau}$ is independent of that of the original edge $\hat{t}$ and is generally different from it as is shown in Fig.1. In the scattering for the disk the modified edge $\hat{\tau}$ coincides with the original edge $\hat{t}$

only two diffraction points D1 and D2.

In MER, the diffracted fields have been empirically calculated as follows:

(a) A local spherical coordinate system with the Z-axis along the modified edge $\hat{\tau}$ is considered at every edge point Q.

(b) The field components of the incident wave on the scatterer are expressed in modified edge coordinate system.

(c) The magnitudes of equivalent edge currents at Q are calculated for the modified edge $\hat{\tau}$ using classical Keller non-uniform expressions, since the new edge satisfies the diffraction law. In this calculation only the radiation terms are included in the incident field components.

(d) The line integration of these currents along the periphery provides the diffracted fields; the direction of equivalent edge currents should be taken along the actual edge $\hat{t}$ and is not along the modified one $\hat{\tau}$.

We will investigate the mathematical principle of MER thus proposed in the following.

Ⅲ. MATHEMATICAL DERIVATION OF MER

(ⅰ) CURVILINEAR COORDINATE SYSTEM ON A CURVED SURFACE

Let S be a curved surface given by the vector form

$$S:\ \mathbf{r} = \mathbf{r}(\sigma, \tau), \tag{3}$$

and let the boundary of S be Γ. We define coordinates of the source, the observer and the any point on S by $P_i(x_i, y_i, z_i)$, $P_o(x_o, y_o, z_o)$ and $Q(x, y, z)$, respectively. Denoting two distances r_i and r_o as

$$r_i = P_i Q \quad \text{and} \quad r_o = P_o Q, \tag{4}$$

we can define the curvilinear coordinate system (σ, τ) which is based on two kinds of trajectories on S as follows:

τ-curves: contour lines such as $r_i + r_o$=const.

σ-curves: the family of orthogonal trajectories to the family of τ-curves.

We hereafter take both parameters, σ and τ, to be equal to arc length along σ, τ -curves. Let us define two unit vectors $\hat{\sigma}$ and $\hat{\tau}$ tangent to τ-curve and σ-curve, respectively, and define the unit vector $\hat{n}$ normal to S. We can normalize $\hat{\tau}$ and $\hat{\sigma}$ to be identical with $\frac{\partial \mathbf{r}}{\partial \tau}$ and $\frac{\partial \mathbf{r}}{\partial \sigma}$, respectively. We also have the orthonormal frame $\{\hat{\sigma}, \hat{n}, \hat{\tau}\}$ on S. Since

$$\frac{\partial(\mathrm{r_i} + \mathrm{r_o})}{\partial \tau} = -\hat{\mathrm{r}}_\mathrm{i} \cdot \hat{\tau} - \hat{\mathrm{r}}_\mathrm{o} \cdot \hat{\tau} = 0 \tag{5}$$

with unit vectors $\hat{r}_i$ and $\hat{r}_o$ oriented toward the source and the observer from any point on S, respectively, we obtain equations

$$(\hat{r}_i + \hat{r}_o) \cdot \hat{\tau} = 0, \tag{6}$$

$$\hat{n} \cdot \hat{\tau} = 0. \tag{7}$$

And we also have the relation

$$\tilde{\nabla}(r_i + r_o) = -(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}\hat{\sigma} + \hat{n}\frac{\partial(r_i + r_o)}{\partial n}, \tag{8}$$

where $\tilde{\nabla} \equiv \hat{\sigma}\frac{\partial}{\partial \sigma} + \hat{n}\frac{\partial}{\partial n} + \hat{\tau}\frac{\partial}{\partial \tau}$.

(ⅱ) LINE INTEGRAL REPRESENTATION

The scattered filed $\mathbf{E}$ for far field due to surface electric current $\mathbf{J}$ and magnetic one $\mathbf{M}$ on S is given by the radiation surface integral

$$\begin{aligned} \mathbf{E} &= j\frac{k\eta}{4\pi}\int_S \hat{r}_o \times (\hat{r}_o \times \mathbf{J})\frac{e^{-jk\,r_o}}{r_o}dS \\ &\quad + j\frac{k}{4\pi}\int_S \hat{r}_o \times \mathbf{M}\frac{e^{-jk\,r_o}}{r_o}dS \end{aligned} \tag{9}$$

where the time factor $e^{j\omega t}$ is assumed and η stands for the intrinsic impedance of free space. We may surface currents $\mathbf{J}$ and $\mathbf{M}$ as $\mathbf{J} = k\mathbf{J}_o e^{-jkr_i}$ and $\mathbf{M} = k\mathbf{M}_o e^{-jkr_i}$, $\mathbf{J}_o$ and $\mathbf{M}_o$ being independent of k.

Next we use the identities for $\hat{r}_o \times (\hat{r}_o \times \mathbf{J}_o)$ and $\hat{r}_o \times \mathbf{M}_o$ as

$$\begin{aligned} \hat{r}_o \times (\hat{r}_o \times \mathbf{J}_o) &= \varsigma_1 \hat{r}_o \times (\hat{r}_o \times \hat{\tau}) \times + \varsigma_2 \, \hat{r}_o \times \hat{\tau} \\ \hat{r}_o \times \mathbf{M}_o &= \varsigma_1' \hat{r}_o \times (\hat{r}_o \times \hat{\tau}) + \varsigma_2' \, \hat{r}_o \times \hat{\tau} \end{aligned} \tag{10}$$

where

$$\varsigma_1 = -\frac{\{\hat{r}_o \times (\hat{r}_o \times \mathbf{J}_o)\} \cdot \hat{\tau}}{1 - (\hat{r}_o \cdot \hat{\tau})^2}, \quad \varsigma_2 = \frac{(\hat{r}_o \times \mathbf{J}_o) \cdot \hat{\tau}}{1 - (\hat{r}_o \cdot \hat{\tau})^2}, \tag{11}$$

$$\varsigma_1' = -\frac{(\hat{r}_o \times \mathbf{M}_o) \cdot \hat{\tau}}{1 - (\hat{r}_o \cdot \hat{\tau})^2} \text{ and } \varsigma_2' = -\frac{\{\hat{r}_o \times (\hat{r}_o \times \mathbf{M}_o)\} \cdot \hat{\tau}}{1 - (\hat{r}_o \cdot \hat{\tau})^2} \tag{12}$$

If the dimension of the scatterer is small enough compared with the wave length, we may write (9) as

$$\begin{aligned} \mathbf{E} \cong\ & jk\eta \hat{r}_o \times (\hat{r}_o \times \mathbf{A}) + jk\eta \hat{r}_o \times \mathbf{B} \\ & + jk\hat{r}_o \times (\hat{r}_o \times \mathbf{A}') \times \hat{r}_o + jk \hat{r}_o \times \mathbf{B}' \end{aligned} \tag{13}$$

$$\mathbf{A} = \frac{k}{4\pi r_o} \int_S \varsigma_1 \hat{\tau} \, e^{-jk r_o} dS \tag{14}$$

$$\mathbf{B} = \frac{k}{4\pi r_o} \int_S \varsigma_2 \hat{\tau} \, e^{-jk r_o} dS \tag{15}$$

$$\mathbf{A}' = \frac{k}{4\pi r_o} \int_S \varsigma_1' \hat{\tau} \, e^{-jk r_o} dS \tag{16}$$

$$\mathbf{B}' = \frac{k}{4\pi r_o} \int_S \varsigma_2' \hat{\tau} \, e^{-jk r_o} dS \tag{17}$$

Using the relation

$$\begin{aligned} &\tilde{\nabla} e^{-jkr_o} \times \hat{n} \\ &\quad = jke^{-jkr_o} (\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma} \, \hat{\tau} \end{aligned} \tag{18}$$

we obtain the significant equation

$$\begin{aligned} \varsigma_1 \hat{\tau} e^{-jk r_o} &= \tilde{\nabla} \left\{ \frac{\varsigma_1 e^{-jk r_o}}{jk(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}} \right\} \times \hat{n} \\ &\quad - \tilde{\nabla} \left\{ \frac{\varsigma_1}{jk(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}} \right\} e^{-jk r_o} \times \hat{n} \end{aligned} \tag{19}$$

Moreover, we note that the relation for any scalar function f

$$\tilde{\nabla} f \times \hat{n} = \nabla f \times \hat{n} \tag{20}$$

holds generally, where $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$. Then (14) can resolved into the sum of two integrals as follows:

$$\begin{aligned} \mathbf{A} = & \frac{k}{4\pi r_o} \int_S \nabla \left\{ \frac{\lambda e^{-jk r_o}}{jk(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}} \right\} \times \hat{n} \, dS \\ & - \frac{k}{4\pi r_o} \int_S \nabla \left\{ \frac{\lambda}{jk(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}} \right\} e^{-jk r_o} \times \hat{n} \, dS \end{aligned} \tag{21}$$

Assuming the case of $(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma} \neq 0$, which corresponds to the case where stationary phase points doesn't exist on S, we can apply Stokes's theorem of the vector form

$$\int_S \nabla f \times \hat{n} \, dS = -\oint_\Gamma f \, d\mathbf{r} \tag{22}$$

to the second integral in (21). Then we obtain

$$\begin{aligned} \mathbf{A} = & -\frac{1}{4\pi r_o} \oint_\Gamma \frac{\varsigma_1 e^{-jk r_o}}{j(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}} \hat{t} \, dl \\ & - \frac{1}{4\pi r_o} \int_S \nabla \left\{ \frac{\varsigma_1}{j(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}} \right\} e^{-jk r_o} \times \hat{n} \, dS \end{aligned} \tag{23}$$

where $\hat{t}$ is the unit tangent vector of the boundary Γ. Next we evaluate the first term of (23) asymptotically. When both the source and the observer are far from the scatterer as the special case, the value of the gradient part in the first integral in (23) is almost equal to zero. In more general case where the source is close to the scatterer, we will adopt a asymptotic technique as following:

$$\begin{aligned} & -\frac{1}{4\pi r_o} \int_S \nabla \left\{ \frac{\varsigma_1}{j(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}} \right\} e^{-jk r_o} \times \hat{n} \, dS \\ & \approx \frac{1}{k} \oint_\Gamma \frac{1}{(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}} \nabla \left\{ \frac{\varsigma_1}{j(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}} \right\} e^{-jk r_o} \times \hat{n} \frac{\partial \tau}{\partial l} dl \end{aligned} \tag{24}$$

For the lowest order k^0, the first term can be ignored and (23) finally becomes

$$\mathbf{A} \cong \frac{1}{4\pi r_o} \oint_\Gamma \mathbf{I}_e \, e^{-jk r_o} \, dl, \tag{25}$$

where $\mathbf{I}_e$ is the electric equivalent edge current,

$$\mathbf{I}_e = \frac{\{\hat{r}_o \times (\hat{r}_o \times \mathbf{J}_o)\} \cdot \hat{\tau}}{jk\,(1 - (\hat{r}_o \cdot \hat{\tau})^2)(\hat{r}_i + \hat{r}_o) \cdot \hat{\sigma}} e^{-jkr_i} \hat{t}. \tag{26}$$

In the similar way we obtain other expressions:

$$\mathbf{B} \cong -\frac{1}{4\pi r_o} \oint_{\Gamma} \frac{(\hat{r}_o \times \mathbf{J}_o)\cdot\hat{\tau}}{jk\,(1-(\hat{r}_o\cdot\hat{\tau})^2)\,(\hat{r}_i+\hat{r}_o)\cdot\hat{\sigma}}\,\hat{t}\,e^{-jk(r_i+r_o)}\,dl$$

$$\mathbf{A}' \cong \frac{1}{4\pi r_o} \oint_{\Gamma} \frac{(\hat{r}_o \times \mathbf{M}_o)\cdot\hat{\tau}}{jk\,(1-(\hat{r}_o\cdot\hat{\tau})^2)\,(\hat{r}_i+\hat{r}_o)\cdot\hat{\sigma}}\,\hat{t}\,e^{-jk(r_i+r_o)}\,dl$$

$$\mathbf{B}' \cong \frac{1}{4\pi r_o} \oint_{\Gamma} \frac{\{\hat{r}_o \times (\hat{r}_o \times \mathbf{M}_o)\}\cdot\hat{\tau}}{jk\,(1-(\hat{r}_o\cdot\hat{\tau})^2)\,(\hat{r}_i+\hat{r}_o)\cdot\hat{\sigma}}\,\hat{t}\,e^{-jk(r_i+r_o)}\,dl \qquad (27)$$

The integrands except e^{-jkr_o} in (27) are identical with electric or magnetic equivalent edge currents in MER. Since the coefficients of $\hat{t}$ in (26) and (27) are scalar, those are invariant for any coordinate system which we adopt. If there are no stationary points on S, the expression (13) with (25), (26) and (27) gives the diffracted fields from the scatterer S. If not so, the contributions from the stationary phase point should be added to (13). It should be also noted that the equivalent edge currents are directly derived by using Stokes's theorem. MER therefore could be applied to scatterers with curved surface to which exact reduction theories[2][3] can't be applicable.

Ⅳ. NUMERICAL RESULTS

Fig.2 is the geometry of the scattreing system used for evaluating the accuracy of MER (25)-(27), in which the scatterer S is the square plate of dimensions 4λ by 4λ (λ being the wavelength) and the electric dipole source with the moment (1,1,1) is located on (λ, λ, 2λ). As surface currents on S due to incident fields $\mathbf{E}^i$ and $\mathbf{H}^i$, we assume the following generalized PO currents

$$\mathbf{J} = (1+\alpha)\,\hat{n}\times\mathbf{H}^i \text{ and } \mathbf{M} = (1-\alpha)\,\mathbf{E}^i \times \hat{n}. \qquad (28)$$

Fig.3 shows the scattering pattern in $\phi_o = 0°$-plane in the usual $r\theta\phi$-spherical coordinate system when $\alpha = -1, 0$. The fine agreements for diffracted fields are obtained between the two methods of calculation.

Ⅴ. CONCLUSION

This paper mathematically derive MER for the first time. MER gives the fine agreement with the generalized PO surface integral for arbitrary position of a source under the far field condition. It is also found that MER could be applied to scatterers with curved surfaces.

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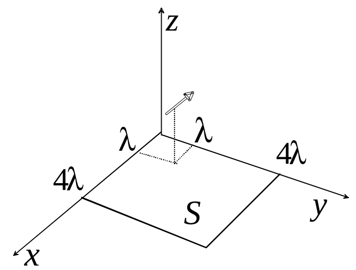

Fig.2 Geometry of the scattering system

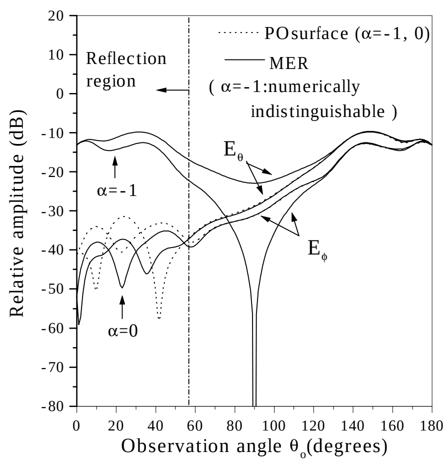

Fig.3 Scattered fields in E_θ and E_ϕ from the square plate