LINE EQUATIONS IN CHAIN-MATRIX EXPRESSION FOR A SINGLE BENT LINE AND TWO BENT PARALLEL LINES

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1. Introduction

Trace lines in printed circuit boards (PCB) are reaching higher and higher density requiring the use of various trace layouts. Line traces are considered to be a sort of transmission line so that in high density PCBs, multi-conductor transmission lines are typical. In addition, some of these are occasionally bent in an arbitrary direction. Coupling phenomenon between traces lines or transmission lines is known as crosstalk in the field of electromagnetic compatibility (EMC). The problem of crosstalk between neighboring traces is a serious issue for signal integrity (SI) in high-speed digital circuits. The so-called telegrapher's equations that deal with line voltage and current are used to analytically investigate crosstalk. Network functions, such as chain matrix, impedance matrix, admittance matrix, etc., are derived from solutions to telegrapher's equations. To derive multi-conductor transmission lines, these equations are expanded into a matrix form. In these cases, the lines should be parallel in a section being considered. However, many parallel bent lines are not equal in length. This presents a difficulty when applying ordinary line equations to bent lines.

In this paper, our final object is to find a line equation for two bent parallel lines. For our purposes, we assume the transmission lines are weakly coupled. Therefore, we assume that each line functions as an isolated line, but that a coupling phenomenon is generated by electromagnetic fields caused by the proximity of a neighboring line. When external electromagnetic fields arrive at a transmission line, induced current flows in that line. This phenomenon is expressed in the form of distributed voltage and current sources along the line. Therefore, telegrapher's equations having forcing terms, nonhomogeneous differential equations, are adopted to analyze the phenomenon [1]-[3]. A solution to modified telegrapher's equations can be obtained by a state variable technique [3]-[4]. We apply this technique to our topic. The electromagnetic fields due to a transmission line can be written with vector potentials. Therefore, we obtain the vector potentials in terms of the line terminal voltages and currents so that the resultant function is derived in a chain-matrix expression.

2. Line equation for a single bent line

We suppose a lossless transmission line installed at height of $z = h$ in the y direction above a perfectly conducting ground plane. To make the line model simple, the medium of the line system is in free space. When the transmission line is excited by external electromagnetic fields, an induced current flows in it. This phenomenon is referred to as coupling of electromagnetic fields to transmission lines. Corresponding to the telegrapher's equations, modified telegrapher's equations can be derived from Maxwell's equations by assuming a propagating wave in the transmission line as a transverse electromagnetic (TEM) mode [1]-[3]. A solution to the modified telegrapher's equations for the line of length ℓ can be as follows,

$$\begin{bmatrix} V(0) \\ I(0) \end{bmatrix} = [F(\ell)] \begin{bmatrix} V(\ell) \\ I(\ell) \end{bmatrix} + \int_0^{\ell} [F(y)] \begin{bmatrix} V_f \\ I_f \end{bmatrix} dy \tag{1}$$

where $[F(y)]$ is a chain matrix or ABCD matrix of the transmission line. Terms V_f and I_f represent effects due to the external fields and correspond to the magnetic field and the electric field contributions, respectively. By using the vector potential $\boldsymbol{A}(A_x, A_y, A_z)$ of the external fields, V_f and I_f can be written as below [5],[6].

$$\begin{bmatrix} V_f \\ I_f \end{bmatrix} = \begin{bmatrix} -j\omega \displaystyle\int_0^h \left\{ \frac{\partial A_z}{\partial y} - \frac{\partial A_z}{\partial y} \right\} dz \\ j\omega C \left\{ -j\omega \displaystyle\int_0^h A_z dz + \frac{1}{j\omega\mu_0\epsilon_0} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \Bigg|_{z=0}^{z=h} \right\} \end{bmatrix} \tag{2}$$

We consider two parallel transmission lines with weak coupling of less than -20 dB, for example. Generally speaking, to obtain a set of solutions to the telegrapher's equations for the two parallel transmission

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lines from a modal analysis, we should consider two orthogonal or independent modes. For two transmission lines, the most popular modes are balanced and unbalanced modes in a transmission-line field, odd and even modes in a microwave-circuit field, or differential and common modes in an EMC field. Thus, the distinctive property characterizing a transmission-line system, characteristic impedance, should be defined as one in one of these modes. Here, we consider a weak coupling; this suggests that the characteristic impedance of each transmission line can be approximated in the same way as that for an isolated line. Therefore, we can apply the coupling concept for external fields of transmission lines to the coupling between two parallel transmission lines. For simplicity, let the both lines be of the same height $z = h$ and same length ℓ and separated by $2w$ as shown in Fig. 1 (a).

First, we consider the equation for line ♯1. That is, we discuss the second term on the right side of (1). The vector potential of line ♯2 is only the y component by virtue of taking into account the image component as

$$A_y = \frac{\mu_0}{4\pi}\left\{\int_0^\ell \frac{I_2(y')e^{-jkr_1}}{r_1}dy' - \int_0^\ell \frac{I_2(y')e^{-jkr_1'}}{r_1'}dy'\right\} \tag{3}$$

where $I_2(y)$ denotes the current flowing in line ♯2 as

$$I_2(y) = -j\frac{V_2(0)}{Z_0}\sin\beta y + I_2(0)\sin\beta y = j\frac{V_2(\ell)}{Z_0}\sin\beta(\ell - y) + I_2(\ell)\cos\beta(\ell - y), \tag{4}$$

and r_1 and r_1' are distances from the current concerned at $y = y'$ to the observing point (x, y, z) as

$$r_1 = \sqrt{(x-w)^2 + (y-y') + (z-h)^2}, \qquad r_1' = \sqrt{(x-w)^2 + (y-y') + (z+h)^2}. \tag{5}$$

By applying (3) into (2), we can obtain the second term on the right side of (1). Moreover, taking into account (4) leads the following equation in a matrix form:

$$\begin{bmatrix} V_1(0) \\ I_1(0) \end{bmatrix} = \begin{bmatrix} \cos\beta\ell & jZ_0\sin\beta\ell \\ j\frac{1}{Z_0}\sin\beta\ell & \cos\beta\ell \end{bmatrix} \begin{bmatrix} V_1(\ell) \\ I_1(\ell) \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} V_2(\ell) \\ I_2(\ell) \end{bmatrix}. \tag{6}$$

A similar procedure can be used for line ♯2 so that we can obtain the resultant line equation in a chain matrix form,

$$\begin{bmatrix} V_1(0) \\ V_2(0) \\ I_1(0) \\ I_2(0) \end{bmatrix} = \begin{bmatrix} \cos\beta\ell & a_2 & jZ_0\sin\beta\ell & b_2 \\ a_1 & \cos\beta\ell & b_1 & jZ_0\sin\beta\ell \\ j\frac{1}{Z_0}\sin\beta\ell & c_2 & \cos\beta\ell & d_2 \\ c_1 & j\frac{1}{Z_0}\sin\beta\ell & d_1 & \cos\beta\ell \end{bmatrix} \begin{bmatrix} V_1(\ell) \\ V_2(\ell) \\ I_1(\ell) \\ I_2(\ell) \end{bmatrix} \tag{7}$$

Note that in many cases considering the terminal mechanical structure, e. g., *riser*, is preferable. That is, A_z components due to the terminal currents should be taken into account in (2).

Next, we consider a bent line as shown in Fig. 1 (b). In the figure, a part of the transmission line of length, ℓ_1, is depicted in the x-y-z coordinate and another part of length, ℓ_2, is in the X-Y-Z coordinate.

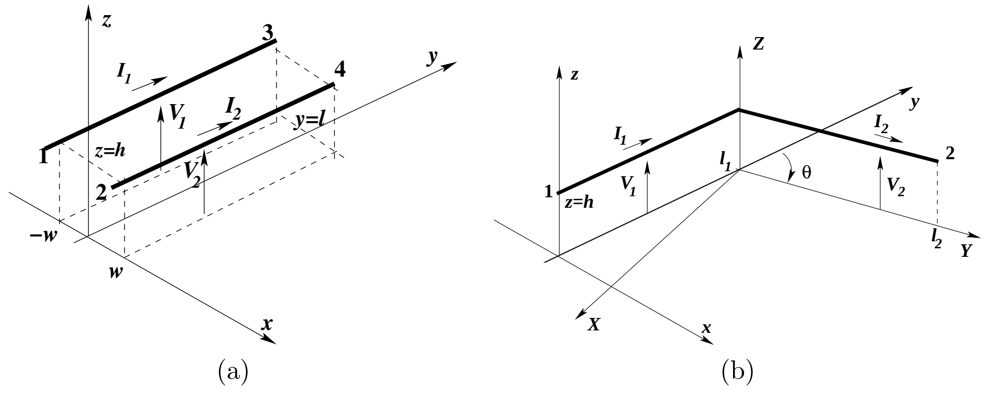


Figure 1: Line models: (a) is for two parallel lines and (b) for a single bent line.

In addition, the angle between the y and the Y axes is θ. Then, the relation between two coordinate systems is

$$\begin{bmatrix} x \\ y-\ell_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}. \tag{8}$$

Two line sections generate the y and Y components of the vector potentials, respectively, so that the components affecting each line section are

$$A_{x2} = A_{Y2}\sin\theta \qquad A_{y2} = A_{Y2}\cos\theta, \qquad \text{and} \qquad A_{X1} = -A_{y2}\sin\theta \qquad A_{Y1} = A_{y1}\cos\theta. \tag{9}$$

For the line section of length ℓ_1, the effects due to the line section of length ℓ_2 are

$$V_{f2} = j\omega\int_0^h \cos\theta\frac{\partial A_{Y2}}{\partial z}dz = j\omega\cos\theta A_{Y2}\Big|_{z=0}^{z=h}, \qquad I_{f2} = \frac{C}{\mu_0\epsilon_0}\left(\sin\theta\frac{\partial A_{Y2}}{\partial x} + \cos\theta\frac{\partial A_{Y2}}{\partial y}\right)\Big|_{z=0}^{z=h} \tag{10}$$

where we can write A_{Y2} as follows by setting the line voltage and current in the line section of length ℓ_2 be $V_2(\cdot)$ and $I_2(\cdot)$:

$$\begin{aligned} A_{Y2} = \frac{\mu_0}{4\pi}\Bigg[\frac{j}{Z_0}&\left\{\int_0^{\ell_2}\frac{\sin\beta(\ell_2-Y')}{R_{12}}e^{-jkR_{12}}dY' - \int_0^{\ell_2}\frac{\sin\beta(\ell_2-Y')}{R'_{12}}e^{-jkR'_{12}}dY'\right\}V_2(\ell_2) \\ &+\left\{\int_0^{\ell_2}\frac{\cos\beta(\ell_2-Y')}{R_{12}}e^{-jkR_{12}}dY' - \int_0^{\ell_2}\frac{\cos\beta(\ell_2-Y')}{R'_{12}}e^{-jkR'_{12}}dY'\right\}I_2(\ell_2)\Bigg] \end{aligned} \tag{11}$$

where $R_{12} = \sqrt{X^2+(Y-Y')^2+(Z-h)^2}$, and $R'_{12} = \sqrt{X^2+(Y-Y')^2+(Z+h)^2}$.

Similar equations for the line section of length ℓ_2 hold so that we resultantly obtain the line equation in a form as

$$\begin{bmatrix} V_1(0) \\ I_1(0) \end{bmatrix} = \begin{bmatrix} A(\ell_1) & B(\ell_1) \\ C(\ell_1) & D(\ell_1) \end{bmatrix}\begin{bmatrix} V_1(\ell_1) \\ I_1(\ell_1) \end{bmatrix} + \begin{bmatrix} a_{2b} & b_{2b} \\ c_{2b} & d_{2b} \end{bmatrix}\begin{bmatrix} V_2(\ell_2) \\ I_2(\ell_2) \end{bmatrix} \tag{12}$$

and

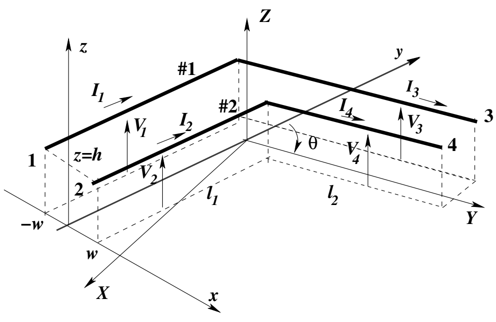
$$\begin{bmatrix} V_2(0) \\ I_2(0) \end{bmatrix} = \begin{bmatrix} A(\ell_2) & B(\ell_2) \\ C(\ell_2) & D(\ell_2) \end{bmatrix}\begin{bmatrix} V_2(\ell_2) \\ I_2(\ell_2) \end{bmatrix} + \begin{bmatrix} a_{1b} & b_{1b} \\ c_{1b} & d_{1b} \end{bmatrix}\begin{bmatrix} V_1(\ell_1) \\ I_1(\ell_1) \end{bmatrix}. \tag{13}$$

From the above equations, considering the continuity relation at the bent point, $y=\ell_1$ and $Y=0$, that is, $V_1(\ell_1) = V_2(0)$ and $I_1(\ell_1) = I_2(0)$, we obtain the line equation in a chain-matrix form for a bent line:

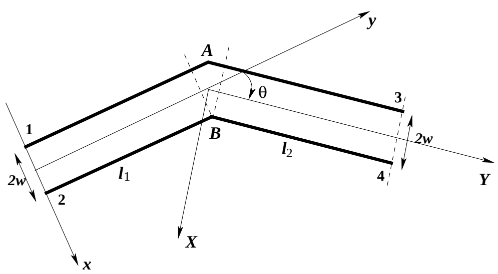
$$\begin{bmatrix} V_1(0) \\ I_1(0) \end{bmatrix} = \left\{\begin{bmatrix} A(\ell_1) & B(\ell_1) \\ C(\ell_1) & D(\ell_1) \end{bmatrix}\begin{bmatrix} 1-a_{1b} & -b_{1b} \\ -c_{1b} & 1-d_{1b} \end{bmatrix}^{-1}\begin{bmatrix} A(\ell_2) & B(\ell_2) \\ C(\ell_2) & D(\ell_2) \end{bmatrix} + \begin{bmatrix} a_{2b} & b_{2b} \\ c_{2b} & d_{2b} \end{bmatrix}\right\}\begin{bmatrix} V_2(\ell_2) \\ I_2(\ell_2) \end{bmatrix}. \tag{14}$$

3. Line equation for two bent parallel lines

Now, we consider a bent-parallel-line model as shown in Fig. 2. The distance between two lines is $2w$ before and after the bent point, and the bent points are denoted with A and B, respectively, in the figure.



(a)



(b)

Figure 2: Model of two bent parallel lines: (a) is for bird's eye view and (b) for top view.

Line $\sharp 1$ is of length ℓ_1 from point 2 to point B plus ℓ_2 from point B to terminal 4. The angle of two line directions, axes y and Y, is θ. This model is often seen in PCBs, but to discuss the line equation strictly from the view of ordinary transmission-line theory is difficult. From a glance at the figure, we notice that the parallel sections are of the same length but excess line sections of different length exist. These excess line sections make analyzing such bent lines difficult.

Here, we apply the method for two parallel lines and a single bent line mentioned earlier to our topic. The two-dimensional coordinates of the bent points are

$$\mathrm{A}\left\{\begin{array}{l} x_A = -w \\ y_A = \ell_1 + 2w\dfrac{1-\cos\theta}{\sin\theta} \end{array}\right. \quad \left\{\begin{array}{l} X_A = -w \\ Y_A = -w\dfrac{1-\cos\theta}{\sin\theta} \end{array}\right. , \quad \mathrm{B}\left\{\begin{array}{l} x_B = w \\ y_B = \ell_1 \end{array}\right. \quad \left\{\begin{array}{l} X_B = w \\ Y_B = w\dfrac{1-\cos\theta}{\sin\theta} \end{array}\right. .$$

And for the right sides of the lines,

$$\mathrm{C}(X,Y) = \left(-w, \ell_2 + w\frac{1-\cos\theta}{\sin\theta}\right), \qquad \mathrm{D}(X,Y) = \left(w, \ell_2 + w\frac{1-\cos\theta}{\sin\theta}\right).$$

Considering the vector potentials due to the currents in the four sections, we can derive the following equations in a general form:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ A_1 & D_1 \end{bmatrix} \begin{bmatrix} V_A \\ I_A \end{bmatrix} + \begin{bmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{bmatrix} \begin{bmatrix} V_B \\ I_B \end{bmatrix} + \begin{bmatrix} a_{13} & b_{13} \\ c_{13} & d_{13} \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} + \begin{bmatrix} a_{14} & b_{14} \\ c_{14} & d_{14} \end{bmatrix} \begin{bmatrix} V_4 \\ I_4 \end{bmatrix} \quad (15)$$

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ A_2 & D_2 \end{bmatrix} \begin{bmatrix} V_B \\ I_B \end{bmatrix} + \begin{bmatrix} a_{21} & b_{21} \\ c_{21} & d_{21} \end{bmatrix} \begin{bmatrix} V_A \\ I_A \end{bmatrix} + \begin{bmatrix} a_{23} & b_{23} \\ c_{23} & d_{23} \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} + \begin{bmatrix} a_{24} & b_{24} \\ c_{24} & d_{24} \end{bmatrix} \begin{bmatrix} V_4 \\ I_4 \end{bmatrix} \quad (16)$$

$$\begin{bmatrix} V_A \\ I_A \end{bmatrix} = \begin{bmatrix} A_3 & B_3 \\ A_3 & D_3 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} + \begin{bmatrix} a_{31} & b_{31} \\ c_{31} & d_{31} \end{bmatrix} \begin{bmatrix} V_A \\ I_A \end{bmatrix} + \begin{bmatrix} a_{32} & b_{32} \\ c_{32} & d_{32} \end{bmatrix} \begin{bmatrix} V_B \\ I_B \end{bmatrix} + \begin{bmatrix} a_{34} & b_{34} \\ c_{34} & d_{34} \end{bmatrix} \begin{bmatrix} V_4 \\ I_4 \end{bmatrix} \quad (17)$$

$$\begin{bmatrix} V_B \\ I_B \end{bmatrix} = \begin{bmatrix} A_4 & B_4 \\ A_4 & D_4 \end{bmatrix} \begin{bmatrix} V_4 \\ I_4 \end{bmatrix} + \begin{bmatrix} a_{41} & b_{41} \\ c_{41} & d_{41} \end{bmatrix} \begin{bmatrix} V_A \\ I_A \end{bmatrix} + \begin{bmatrix} a_{42} & b_{42} \\ c_{42} & d_{42} \end{bmatrix} \begin{bmatrix} V_B \\ I_B \end{bmatrix} + \begin{bmatrix} a_{43} & b_{43} \\ c_{43} & d_{43} \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix} \quad (18)$$

A chain matrix expression for terminals $1, 2, 3,$ and 4 can be obtained by combining the equations above.

4. Conclusion

We derived a line equation expressed in a chain matrix for parallel bent lines by applying the concept of the coupling of external fields to transmission lines. The equations derived here are for weak coupling. In future, we need to check the effectiveness of the equation through experiment, and extend it to PCB traces.

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